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# Gravitational collapse of a cylindrical null shell in vacuum

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**Abstract**

Barrabès-Israel null shell formalism is used to study the gravitational collapse of a thin cylindrical null shell in vacuum. In general the lightlike matter shell whose history coincides with a null hypersurface is characterized by a surface energy density. In addition, a gravitational impulsive wave is present on this null hypersurface whose generators admit both the shear and expansion. In the case of imposing the cylindrical flatness the surface energy-momentum tensor of the matter shell on the null hypersurface vanishes and the null hyper- surface is just the history of the gravitational wave.

**Keywords:** general relativity, null shell, gravitational waves

## 1. Introduction

Many practical problems of general relativity and cosmology involve idealized models constructed by gluing two regions with different metrics across a hypersurface or thin shell having a $\delta$-function singularity in its Riemann tensor due to the discontinuity in the metric's transverse derivative across the shell. The description of timelike (or spacelike) thin shells is well known within general relativity since the outstanding work of Israel [1]. Later, an extension of the Israel formalism to the null or lightlike case was presented by Barrabès and Israel [2]. Recently, Poisson has introduced a user-friendly reformulation of the Barrabès-Israel original work together with an illustration of the formalism [3]. On the other hand, relativistic dynamics of cylindrical symmetric thin shells as sources of gravitational field have been studied during the whole development of general relativity. A number of papers have been concerned with a rotating cylindrical shell in general relativity. In particular, by studying the collapse of a cylindrical shell made of pressure-free counter-rotating particles in vacuum, the authors realized that the rotation always halts the collapse [4]. Other examples of cylindrical thin shells were studied by the authors in Refs [5,6,7] who considered a cylindrical thin shell separating an interior flat spacetime from an exterior radiation-filled curved spacetime. In addition, the various kinds of shell sources for static Levi-Civita and Lewis spacetime respectively have been discussed in Refs [8, 9]. Charged generalization of the Levi-Civita spacetime and their shell sources have been studied in [10].

In this paper, we study the dynamics of a collapsing cylindrical null shell immersed in vacuum. For this purpose we use Barrabès-Israel (BI) null shell formalism [2] to investigate the matching and find the junction conditions. Section 2 is devoted to the formulation of the problem and the junction conditions. In section 3 we investigate the case of gravitational wave. A conclusion follows then in section 4.

Natural geometrized units, in which G=c=1 are used throughout the paper. The null hypersurface is denoted by $\Sigma$ . The symbol $\big|_{\Sigma}$ means "evaluated on the null hypersurface". Latin indices range over the intrinsic coordinates of $\Sigma$ denoted by $\xi^a$ and Greek indices over the coordinates of the 4-manifolds.

## 2. Null shell formalism

Consider the gravitational collapse of a thin cylindrical shell in vacuum. We imagine that the collapse proceeds at the speed of light such that the history of the shell coincides with a null hypersurface $\Sigma$. Taking spacetime to be flat inside the shell ($M_-$) we write the metric there as

$$ds_-^2 = -dt^2 + dr^2 + dz^2 + r^2 d\phi^2, \tag{1}$$

in terms of the Einstein-Rosen canonical cylindrical coordinates (t,r,z, $\phi$). For the exterior vacuum spacetime of the shell ($M_+$) we take the cylindrical symmetric metric in the general form [4]

$$ds_+^2 = -e^{2(\gamma-\psi)}(dT^2 - dR^2) + e^{2\psi}dz^2 + R^2e^{-2\psi}d\phi^2 \quad (2)$$

expressed in the same coordinates $z$ and $\phi$ but in terms of distinct coordinates T and $R$ in general. Here $\gamma$ and $\psi$ are functions of T and $R$, and for the Einstein field equation $R_{\mu\nu} = 0$ one gets the gravitational wave field [4]

$$\psi_{,TT} - \psi_{,RR} - \frac{\psi_{,R}}{R} = 0, \quad (3)$$

Together with

$$\gamma_{,T} = 2R\, \psi_{,T}\psi_{,R},\ \ \gamma_{,R} = R(\psi^2_{,T} + \psi^2_{,R}), \quad (4)$$

Taking $\xi^a = (\lambda, z, \phi)$ with a=1,2,3 as the intrinsic coordinates on $\Sigma$, and $\lambda$ a parameter on the null generators of the hypersurface we note that the null hypersurface $\Sigma$ as seen from $M_-$ is described by the parametric equations

$$\begin{aligned} t + r = \upsilon_- &= const, \\ r &= -\lambda, \\ z &= z, \\ \theta &= \theta. \end{aligned} \quad (5)$$

since $\lambda$ always increases along the generators for a collapsing hypersurface r decreases along the generators. It is then seen that the induced metric on $\Sigma$ is given by $ds^2_\Sigma = dz^2 + \lambda^2 d\phi^2$.

As seen from $M_+$, one can obtain the description of the hypersurface by solving the Euler-Lagrange equations. Defining

$$2K = -e^{2(\gamma-\psi)}\,\dot{T}^2 + e^{2(\gamma-\psi)}\,\dot{R}^2 = 0, \quad (6)$$

where $\dot{q}$ denotes $\frac{dq}{d\lambda}$, the Euler-Lagrange equations are written as

$$\frac{\partial K}{\partial q} - \frac{d}{d\lambda}\frac{\partial K}{\partial \dot{q}} = 0\ . \quad (7)$$

Solving this equation for $q=T$ one gets

$$2(\gamma-\psi)_{,T}(\dot{T}^2 - \dot{R}^2) - \frac{d}{d\lambda}(2e^{2(\gamma-\psi)}\,\dot{T}) = 0 \quad (8)$$

By virtue of eq. (6) the first term in eq. (8) is zero, so that we arrive at

$$\dot{T} = Ce^{-2(\gamma-\psi)}, \quad (9)$$

where $C > 0$ is an integration constant. In the case that the exterior spacetime $M_+$ approaches to the flat Minkowski metric as $r \to \infty$, we choose $C=1$. Substituting the solution (9) into eq. (6) leads to

$$\dot{R} = -Ce^{-2(\gamma-\psi)} \quad (10)$$

where the minus sign indicates that the shell is collapsing. From Eqs. (9) and (10) we end up with the following parametric equation of the hypersurface $\Sigma$ as seen from $M_+$

$$T + R \equiv \upsilon_+ = const. \quad (11)$$

Now the requirement of the continuity of the induced metric on $\Sigma$ yields the following matching conditions:

$$e^{2\psi} \overset{\Sigma}{=} 1, \qquad r \overset{\Sigma}{=} \mathrm{R}e^{-\psi} \quad (12)$$

For further applications, we note that the differentiation of $\psi(T,R) = 0$ on $\Sigma$ leads to

$$\psi_{,T} \overset{\Sigma}{=} \psi_{,R}\,. \quad (13)$$

We must calculate the tangent basis vectors $e_a = \partial/\partial\xi^a$ on both sides of $\Sigma$. Having written $x_+^\mu$ in terms of $x_-^\mu$ by eq. (12), we get

$$e_\lambda^\mu\Big|_- = (1,-1,0,0)\big|_\Sigma, \quad e_z^\mu\Big|_- = \delta_z^\mu, \quad e_\phi^\mu\Big|_- = \delta_\phi^\mu \quad (14)$$

$$e_\lambda^\mu\Big|_+ = Ce^{-2\gamma}(1,-1,0,0)\big|_\Sigma,\ e_z^\mu\Big|_+ = \delta_z^\mu,\ e_\phi^\mu\Big|_+ = \delta_\phi^\mu \quad (15)$$

Recalling that $\lambda$ is a parameter on the null geodesic generators of $\Sigma$ we choose the tangent-normal vector $n^\mu$ to coincide with the tangent basis vector associated with the parameter $\lambda$, so that $n^\mu = e_\lambda^\mu$. We may then complete the basis by a transverse null vector $N^\mu$ uniquely defined by the four conditions $n_\mu N^\mu = -1$, $N_\mu e_A^\mu = 0 (A = \theta, \phi)$, and $N_\mu N^\mu = 0$. We find

$$N_\mu\big|_- = \frac{1}{2}(-1,+1,0,0)\big|_\Sigma\,, \quad (16)$$

$$N_\mu\big|_+ = \frac{e^{2\gamma}}{2C}(-1,+1,0,0)\big|_\Sigma\,. \quad (17)$$

Furthermore, the induced metric on $\Sigma$ given by $g_{ab} = g_{\mu\nu}e_a^\mu e_b^\nu\big|_\pm$ is computed to be $g_{ab} = diag(0,1,r^2)$, which is the same on both sides of the hypersurface. Defining a pseudo-inverse of the induced metric $g_{ab}$ on $\Sigma$ as $g_*^{ac}g_{bc} = \delta_b^a + n^a N_\mu e_b^\mu$, with $n^a = \delta_\lambda^a$ [2], one gets $g_*^{ab} = diag(0,1,\frac{1}{r^2})$.

The final junction condition is formulated in terms of the jump in the extrinsic curvature. Using the definition $K_{ab} = e_a^\mu e_b^\nu \nabla_\mu N_\nu, \pi$ we may therefore compute the transverse extrinsic curvature tensor [2] on both sides of $\Sigma$. Its non-vanishing components on the minus side are found as

$$K_{\phi\phi}\big|_- = \frac{r}{2}\Big|_\Sigma\,, \quad (18)$$

$$K_{zz}\big|_- = 0\,, \tag{19}$$

$$K_{\lambda\lambda}\big|_- = 0\,, \tag{20}$$

The corresponding non-vanishing components on the plus side are

$$K_{\phi\phi}\big|_+ = \frac{R}{2C}(1-2R\psi_{,T})\Big|_\Sigma\;, \tag{21}$$

$$K_{zz}\big|_+ = \frac{\psi_{,T}}{C}\Big|_\Sigma\;, \tag{22}$$

$$K_{\lambda\lambda}\big|_+ = 0\;. \tag{23}$$

The jump in the transverse extrinsic curvature across the null hypersurface given by $\gamma_{ab} = 2[K_{ab}]$ has the following components:

$$\gamma_{\lambda\lambda} = 0\,, \tag{24}$$

$$\gamma_{\phi\phi} = \frac{R}{C}(1-2R\psi_{,T} - C)\Big|_\Sigma\,, \tag{25}$$

where we have used eq. (13).

$$\gamma_{zz} = \frac{2\psi_{,T}}{C}\Big|_\Sigma\,. \tag{26}$$

The surface energy-momentum tensor of the lightlike shell having the null hypersurface $\Sigma$ as its history is directly related to the jump in the transverse extrinsic curvature. In the tangent basis $e_a$, it can be written in the form [3]

$$S^{ab} = fn^a n^b + pg_*^{ab} + j^a n^b + j^b n^a\,, \tag{27}$$

where

$$f = -\frac{1}{16\pi} g_*^{ab}\gamma_{ab}\,, \tag{28}$$

represents the surface energy density,

$$p = -\frac{1}{16\pi}\gamma_{ab}\, n^a n^b\,, \tag{29}$$

displays the isotropic surface pressure, and

$$j^a = -\frac{1}{16\pi} g_*^{ac}\gamma_{cd}\, n^d \tag{30}$$

represents the surface current of the lightlike shell. All these surface quantities are measured by a family of freely-moving observers crossing the null hypersurface. Using the jumps in the extrinsic curvature obtained above, we notice first that the surface current term vanishes identically. The energy density and pressure are then calculated as

$$\begin{aligned} f &= -\frac{1}{16\pi}\left(\frac{\gamma_{\phi\phi}}{r^2} + \gamma_{zz}\right), \\ &= \frac{C-1}{16\pi C\, r}\Big|_\Sigma\,, \end{aligned} \tag{31}$$

$$16\pi p = -\gamma_{\lambda\lambda} = 0\,. \tag{32}$$

Assuming the positivity of the surface energy density of the shell given by eq. (31) we see that the matching condition requires that $C \geq 1$. Therefore we see that in general the energy density $f$ is the only non-vanishing surface quantity due to the presence of a lightlike shell of matter with the history $\Sigma$ collapsing with the speed of light to zero radius. In the special case of imposing the cylindrical flatness (as $r\to\infty, \psi$ and $\gamma\to 0$) the surface energy-momentum tensor of the matter shell on $\Sigma$ vanishes so that the null hypersurface $\Sigma$ would be just a smooth boundary. In addition, the presence of a pressure-free null shell indicates that the null generators are affinely parameterized on both sides of the hypersurface $\Sigma$ [3].

## 3. Null shell and gravitational wave

The presence of gravitational waves having the null hypersurface $\Sigma$ as history is seen in the following way. Let us first construct a null tetrad frame on. Consider now a congruence of timelike geodesics with continuous 4-velocity $u$ across $\Sigma$ so that $[u.u] = [u_\alpha e_a^\alpha] = 0$. On the null hypersurface $\Sigma$ we have the normal $n_-^\mu$, which is tangential to $\Sigma$, and the timelike vector field $u^\mu$, crossing the null hypersurface such that $u^\alpha n_\alpha = -s < 0$. It is then advantageous to introduce on $\Sigma$ a transverse null vector field $l^\mu$ defined by $l^\mu = -\frac{1}{2s^2}n^\mu + \frac{1}{s}u^\mu$, satisfying the normalization condition $l^\alpha n_\alpha = -1$ [11]. Let next $m^\mu$ and $\bar{m}^\mu$ be a complex covariant vector field and its complex conjugate (indicated by a bar) being chosen so that they are null $(m^\alpha m_\alpha = \bar{m}^\alpha \bar{m}_\alpha = 0)$, tangent to $\Sigma$, orthogonal to $n^\mu$ and $l^\mu$ and satisfy $m_\alpha \bar{m}^\alpha = 1$. Now $n_-^\mu$, $l^\mu$, $m^\mu$, and $\bar{m}^\mu$ constitute the desired null tetrad frame on $\Sigma$ which will be used in the following. We get

$$m^\mu = \frac{1}{\sqrt{2}}(0,0,-i,\frac{1}{r})\Big|_\Sigma\,, \tag{33}$$

$$\bar{m}^\mu = \frac{1}{\sqrt{2}}(0,0,i,\frac{1}{r})\Big|_\Sigma\,. \tag{34}$$

Using this null tetrad, the Newman-Penrose component of the singular part of the Weyl tensor of Petrov type N characterizing an impulsive gravitational wave with history $\Sigma$ is calculated as [11]

$$\begin{aligned} \breve{\psi}_4 &= \frac{1}{2}\gamma_{ab}\bar{m}^a \bar{m}^b\,, \\ &= \frac{1}{4r^2}(\gamma_{\phi\phi} - r^2\gamma_{zz})\,, \\ &= \frac{1}{4CR}(1 - C - 4R\psi_{,T})\big|_\Sigma\,, \end{aligned} \tag{35}$$

where we have used eqs. (25), (26) and (34). This shows explicitly that in general a null shell and an impulsive wave co-exist, each with history $\Sigma$ as the worldsheet. In this case induced geometry on $\Sigma$, inherited from the embedding spacetimes, is of type III according to a classification introduced by Penrose [12]. However in the special case that the cylindrical flatness is assumed (*C=1*), the null hypersurface $\Sigma$ is just the history of an impulsive gravitational wave.

The expansion $\theta$ and complex shear $\sigma$ of the geodesic generators of the null hypersurface $\Sigma$ can now be defined by the following relations using the null tetrad [11]:

$$\theta = m^{\mu}\bar{m}^{\nu}\nabla_{\nu}n_{\mu} = -\frac{1}{2R}Ce^{-2\gamma}\Big|_{\Sigma} \tag{36}$$

$$\sigma = m^{\mu}m^{\nu}\nabla_{\nu}n_{\mu} = -\frac{1}{2R}Ce^{-2\gamma}\Big|_{\Sigma} \tag{37}$$

This shows that $\Sigma$ is a future null cone generated by the null geodesics with the expansion and shear as given by eqs. (36) and (37).

## 4. Conclusion

We have studied the relativistic dynamics of a collapsing cylindrical null shell in vacuum. In general the lightlike matter shell which its history coincides with a null hypersurface admits a surface energy density with no surface stress and collapses to zero radius. In addition, a gravitational impulsive wave is present on this null hypersurface whose generators have the shear and expansion. The shell and the wave propagate independently and have the same null hypersurface $\Sigma$ as the worldsheet. We have also shown that if the cylindrical flatness is imposed then there would be no shell and the null hypersurface $\Sigma$ is just the history of an impulsive gravitational wave.

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