

## Thermodynamics of rotating black branes in gravity with first order string corrections

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(Received 16 April 2005 ; in final form 7 October 2005)

### Abstract

In this paper, the rotating black brane solutions with zero curvature horizon of classical gravity with first order string corrections are introduced. Although these solutions are not asymptotically anti de Sitter, one can use the counterterm method in order to compute the conserved quantities of these solutions. Here, by reviewing the counterterm method for asymptotically anti de Sitter spacetimes, the conserved quantities of these rotating solutions are computed. Also a Smarr-type formula for the mass as a function of the entropy and the angular momenta is obtained, and it is shown that the conserved and thermodynamic quantities satisfy the first law of thermodynamics. Finally, a stability analysis in the canonical ensemble is performed, and it is shown that the system is thermally stable. This is in commensurable with the fact that there is no Hawking-Page phase transition for black object with zero curvature horizon.

**Keywords:** black holes, dilaton gravity, counterterm method

### 1. Introduction

It follows directly from the observation of high red-shift supernova [1] and indirectly from the measurement of angular fluctuations [2] of cosmic microwave background fluctuations that, at the present epoch, the universe expands with acceleration instead of deceleration along the scheme of standard Friedmann models. These astrophysical data have created a great deal of attention to the asymptotically non-flat spacetimes. As it is known asymptotically flat spacetimes are the solutions of Einstein equation without cosmological term, and therefore it seems natural to use other consistent theory of gravity with a more general action. This action may be written, for example, through the use of string theory. The effect of string theory on classical gravitational physics is usually investigated by means of a low energy effective action which describes gravity at the classical level [3]. This effective action consists of the Einstein-Hilbert action plus curvature-squared terms and higher powers as well, and in general give rise to fourth order field equations and bring in ghosts. However, if the effective action contains the higher powers of curvature in particular combinations, then only second order field equations are produced and consequently no ghosts arise [4]. The effective action

obtained by this argument is precisely of the form proposed by Lovelock [5]. The string tree level effective action for the massless boson sector with Lovelock term in the unit system  $G_n = c = 1$ , in vacuum may be written as [6]:

$$I_G^{(string)} = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-\bar{g}} e^{-2\Phi} \left( \bar{\mathcal{R}} - \frac{2(n-9)}{3\alpha'} + 4(\nabla\Phi)^2 + \lambda_0\alpha'(\bar{\mathcal{L}}_{GB} + \dots) + O(\alpha'^2) \right), \quad (1)$$

where  $\alpha'$  is the string expansion parameter. Applying a conformal transformation ( $\bar{g}_{\mu\nu} = e^{\zeta\Phi} g_{\mu\nu}$ ) one obtains the action in the Einstein frame

$$I_G^{(Einstein)} = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} \left\{ \mathcal{R} - 2\Lambda e^{\zeta\Phi} - \zeta(\nabla\Phi)^2 + \alpha e^{-\zeta\Phi} [\mathcal{L}_{GB} + \beta(\nabla\Phi)^4] \right\} + O(\alpha'^2), \quad (2)$$

where  $\Lambda = (n-9)/3\alpha'$ ,  $\alpha = \lambda_0\alpha'$ ,  $\beta = (n-3)\zeta^2/(n-1)$ ,  $\zeta = 4/(n-1)$  and  $\mathcal{L}_{GB}$  is the Gauss-Bonnet term,

$$\mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\lambda\kappa}\mathcal{R}^{\mu\nu\lambda\kappa},$$

which is the leading order quantum gravity correction in

the heterotic closed string theory at tree level.  $\mathcal{R}$ ,  $\mathcal{R}_{\mu\nu}$  and  $\mathcal{R}_{\mu\nu\lambda\kappa}$  are the Ricci scalar, Ricci and Riemann tensors of the metric  $g_{\mu\nu}$  of the manifold  $\mathcal{M}$ .

Choosing the Gauss-Bonnet combination for the term quadratic in the curvature requires that one should also consider the term quartic in  $\nabla\Phi$ , as given in eq. (2). It is worthwhile to note that the Gauss-Bonnet term couples to the scalar field  $\Phi$ .

In this paper, I consider the rotating black brane solutions of the action (2) with  $\alpha = 0$ . I also study the phase behavior of the rotating black branes in  $(n+1)$  dimensions with zero curvature horizon and show that there is no Hawking-Page phase transition in spite of the angular momentum of the branes. This is incommensurable with the fact that there is no Hawking-Page phase transition for a black object whose horizon is diffeomorphic to  $\mathbb{R}^p$  ( $p$ -brane solution) and therefore the system is always in the high temperature phase [7].

The outline of this paper is as follows. I give a brief review of the field equations in Sec. 2. In Sec. 3, I review the counterterm method inspired by the AdS/CFT correspondence in order to compute the conserved quantities of asymptotically (A)dS spacetimes, and generalize it in order to find the conserved quantities of the solutions of Einstein-dilaton gravity with zero curvature boundary. In Sec.4, I study the thermodynamics of the black branes, and perform a thermal stability analysis. I finish the paper with some concluding remarks.

## 2. Field equations

Varying the action given in eq. (2) with respect to  $g_{\mu\nu}$  gives the gravitational field equation in vacuum

$$G_{\mu\nu} = -\Lambda e^{\zeta\Phi} g_{\mu\nu} + \zeta \left[ \nabla_{\mu} \Phi \nabla_{\nu} \Phi - \frac{1}{2} (\nabla\Phi)^2 g_{\mu\nu} \right] - 2\alpha e^{\zeta\Phi} \left\{ H_{\mu\nu} + 2P_{\rho\mu\nu\sigma} \left[ \zeta \nabla^{\rho} \nabla^{\sigma} \Phi - \zeta^2 \nabla^{\rho} \Phi \nabla^{\sigma} \Phi \right] + \beta \left[ (\nabla\Phi)^2 \nabla_{\mu} \Phi \nabla_{\nu} \Phi - \frac{1}{4} (\nabla\Phi)^4 g_{\mu\nu} \right] \right\} + O(\alpha^2), \quad (3)$$

where  $G_{\mu\nu}$  is the Einstein tensor,

$$P_{\rho\mu\nu\sigma} = \mathcal{R}_{\rho\mu\nu\sigma} + \mathcal{R}_{\mu\nu} g_{\rho\sigma} + \mathcal{R}_{\rho\sigma} g_{\mu\nu} - \mathcal{R}_{\nu\rho} g_{\mu\sigma} - \mathcal{R}_{\mu\sigma} g_{\nu\rho} + \frac{1}{2} \mathcal{R} (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\nu} g_{\rho\sigma}),$$

is the divergence free part of the Riemann tensor, and

$$H_{\mu\nu} = \mathcal{R} \mathcal{R}_{\mu\nu} - 2\mathcal{R}_{\mu\rho} \mathcal{R}^{\rho}_{\nu} g_{\rho\sigma} - 2\mathcal{R}^{\rho\sigma} \mathcal{R}_{\mu\rho\nu\sigma} + \mathcal{R}_{\mu}^{\rho\sigma\kappa} \mathcal{R}_{\nu\rho\sigma\kappa} - \frac{1}{4} g \mathcal{L}_{GB\mu\nu},$$

is the Gauss-Bonnet tensor.

The equation of motion for the dilaton field  $\Phi$  can be found by the variation of the action (2) with respect to

$\Phi$  as

$$\nabla^2 \Phi - \Lambda e^{\zeta\Phi} - \frac{1}{2} \alpha e^{-\zeta\Phi} \left[ \mathcal{L}_{GB} - 3\beta (\nabla\Phi)^4 \right] + \frac{4\beta}{\zeta} \left[ (\nabla\Phi)^2 \nabla^2 \Phi + 2\nabla^{\rho} \Phi \nabla^{\sigma} \Phi \nabla_{\rho} \nabla_{\sigma} \Phi \right] = 0 \quad (4)$$

The left hand side of eq. (3) is just the Einstein equation in vacuum, while the terms in the right hand side come into play by string theory. In the absence of dilaton field, eq. (3) reduces to the field equation of Einstein-Gauss-Bonnet gravity. Static spherically symmetric black hole solutions of the Gauss-Bonnet gravity were found in Ref. [8], while black hole solutions with nontrivial topology were studied in Refs. [9]. All of these known solutions are static. Recently I introduced two classes of asymptotically anti-de Sitter rotating solutions in the Einstein-Gauss-Bonnet gravity and considered their thermodynamics [10]. Also I considered asymptotically flat, AdS and dS solutions of Gauss-Bonnet gravity without cosmological term [11].

In this paper I consider the field eqs (3) and (4) with  $\alpha = 0$  in vacuum. The static solutions of these field equations have been investigated by many authors. The static spherically symmetric solutions of these eqs with positive, negative and zero curvature horizons have been considered in [12]. I will consider the rotating solutions of these field equations.

## 3. The conserved quantities

The concept of action and energy momentum play central roles in gravity, but as it is known there is no good local notion of energy for a gravitating system. A quasilocal definition of the energy and conserved quantities can be found in [13]. The known obstacle to the straightforward definition of the gravitational action and therefore the conserved quantities of a gravitating system is that the action and therefore all the conserved quantities diverge. An approach toward evaluating them has been to carry out all computations relative to some reference spacetime that is regarded as the ground state for the class of spacetimes of interest. This is done by taking the original action for gravity fields and subtracting from it a reference action  $I_0$ , which is a functional of the induced metric  $\gamma$  on the boundary  $\partial\mathcal{M}$ .

Conserved quantities are then computed relative to this boundary, which can then be taken to (spatial) infinity if desired.

This approach has been widely successful in providing the conserved quantities of the system in regions of both finite and infinite spatial extent. Unfortunately it suffers from several drawbacks. The choice of reference spacetime is not always unique [14], nor is it always possible to embed a boundary with a given induced metric into the reference background. Indeed, for Kerr spacetimes this latter problem forms a serious obstruction towards calculating the subtraction energy, and calculations have only been performed in the slow-rotating regime [15].

An extension of this approach was developed for asymptotically AdS spacetimes based on the conjecture AdS/CFT correspondence [16]. Since quantum field theories in general contain counterterms, it is natural from the AdS/CFT viewpoint to append a boundary term  $I_{ct}$  to the action which depends on the intrinsic geometry of the (timelike) boundary at large spatial distances. This requirement, along with general covariance, implies that these terms be functionals of curvature invariants of the induced metric and has no dependence on the extrinsic curvature of the boundary. An algorithmic procedure [17] exists for constructing  $I_{ct}$  for asymptotically AdS spacetimes, and so its determination is unique. Addition of  $I_{ct}$  will not affect the bulk equations of motion, thereby eliminating the need to embed the given geometry in a reference spacetime. Hence conserved quantities can now be calculated intrinsically for any given spacetime.

The AdS/CFT correspondence is now a fundamental concept which furnishes a means for calculating the action and conserved quantities intrinsically without reliance on any reference spacetime [18]. This conjecture has been recently extended to the case of asymptotically de Sitter spacetimes [19, 20]. Although the (A)dS/CFT correspondence applies for the case of spatially infinite boundary, it was also employed for the computation of the conserved and thermodynamic quantities in the case of a finite boundary [21]. This conjecture has also been used for the case of black objects with nonspherical horizons [22]. Here, for completeness I give a brief review of the counterterm method inspired by the (A)dS/CFT correspondence.

If the manifold  $\mathcal{M}$  has a boundary  $\partial\mathcal{M}$ , then variational principle of the action (2) on the boundary is ill-defined. In order to have a well-defined variational principle on the boundary for the case of  $\alpha=0$ , one should add the Gibbons Hawking boundary term

$$I_b = \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-\gamma} K(\gamma), \quad (5)$$

to the action (2), where  $K(\gamma)$  is the trace of the extrinsic curvature  $K^{\mu\nu}$  of any boundary(ies)  $\partial\mathcal{M}$  of the manifold  $\mathcal{M}$ , with induced metric(s)  $\gamma_{ij}$ . In general the action (2) and (5) are both divergent when evaluated on the solutions, as is the Hamiltonian, and other associated conserved quantities. The AdS/CFT correspondence states that if the metric near the conformal boundary ( $x \rightarrow 0$ ) can be expanded in the asymptotically AdS form,

$$ds^2 = \frac{dx^2}{l^2 x^2} + \frac{1}{x^2} \gamma_{ij} dx^i dx^j, \quad (6)$$

with nondegenerate metric  $\gamma_{ij}$ , then one may remove the divergent terms in the action by adding a counterterm action,  $I_{ct}$ , which is a functional of the boundary curvature invariants. The counterterm for

asymptotically AdS spacetimes up to seven dimensions is

$$I_{ct} = \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-\gamma} \left\{ \frac{n-1}{l} - \frac{l\Theta(n-3)}{2(n-2)} R - \frac{l^3 \Theta(n-5)}{2(n-4)(n-2)^2} \left( R_{ij} R^{ij} - \frac{n}{4(n-1)} R^2 \right) + \dots \right\}, \quad (7)$$

where  $R, R_{ijkl}$ , and  $R_{ij}$  are the Ricci scalar, Riemann and Ricci tensors of the boundary metric  $\gamma_{ij}$ , and  $\Theta(x)$  is the step function which is equal to one for  $x \geq 0$  and zero otherwise. Although other counterterms (of higher mass dimension) may be added to  $I_{ct}$ , they will make no contribution to the evaluation of the action or Hamiltonian due to the rate at which they decrease toward infinity.

Of course, for even  $n$  one has logarithmic divergences in the partition function which can be related to the Weyl anomalies in the dual conformal field theory [23]. These logarithmic divergences associated with the Weyl anomalies of the dual field theory for  $n=4$  and  $n=6$  are [24]:

$$I_{log} = -\frac{\ln \varepsilon}{64\pi^3} \int d^4 x \sqrt{-\gamma^0} \left( R_{ij}^0 R^{(0)ij} - \frac{1}{3} (R^0)^2 \right) \quad (8)$$

$$I_{log} = \frac{\ln \varepsilon}{8^4 \pi^3} \int d^6 x \sqrt{-\gamma^0} \left\{ \frac{3}{50} (R^0)^3 + R_{ijkl}^0 R^{(0)ij} R^{(0)kl} - \frac{1}{2} R^0 R_{ij}^0 R^{(0)ij} + \frac{1}{5} R^{(0)ij} D_i D_j R^0 - \frac{1}{2} R^{(0)ij} D_k D^k R_{ij}^0 + \frac{1}{20} R^0 D_k D^k R^0 \right\}. \quad (9)$$

In eqs. (8) and (9)  $R_0$  and  $R_{ij}^0$  are the Ricci scalar and Ricci tensor of the leading order metric  $\gamma^0$  in the following expansion:

$$\gamma_{ij} = \gamma_{ij}^0 + x^2 \gamma_{ij}^2 + x^4 \gamma_{ij}^4 + \dots, \quad (10)$$

and  $D_i$  is the covariant derivative constructed by the leading order metric  $\gamma^0$ .

Having the total finite action

$$I = I_G + I_b + I_{ct} \quad (11)$$

one can use the Brown and York definition [13] to construct a divergence free stress-energy tensor as

$$T^{ij} = \frac{1}{8\pi} \left\{ (K^{ij} - K\gamma^{ij}) - \frac{n-1}{l} \gamma^{ij} + \frac{l\Theta(n-3)}{(n-2)} (R^{ij} - \frac{1}{2} R\gamma^{ij}) - \frac{l^3 \Theta(n-5)}{2(n-4)(n-2)^2} \left[ \frac{1}{2} (R_{kl} R^{kl} - \frac{n}{4(n-1)} R^2) \gamma^{ij} + \frac{n}{2(n-1)} R R^{ij} - 2R_{kl} R^{ijkl} + \frac{n-2}{4(n-1)} \nabla^i \nabla^j R - \nabla^2 R^{ij} + \frac{1}{2(n-1)} \gamma^{ij} \nabla^2 R \right] + \dots \right\}, \quad (12)$$

The above stress-tensor is divergence free for  $n=6$ , but one can always add more counterterms to have a finite action in higher dimensions (see e.g. [17]).

To compute the conserved charges of the spacetime, one should choose a spacelike surface  $\mathcal{B}$  in  $\partial\mathcal{M}$  with metric  $\sigma_{ab}$ , and write the boundary metric in ADM form:

$$\gamma_{ij}dx^i dx^j = -Ndt^2 + \sigma_{ab}(d\varphi^i + V^i dt)(d\varphi^j + V^j dt),$$

where the coordinates  $\varphi^i$ 's are the angular variables parameterizing the hypersurface of constant  $t$  around the origin, and  $N$  and  $V^i$  are the lapse and shift functions respectively. When there is a Killing vector field  $\xi$  on the boundary, then the quasilocal conserved quantities associated with the stress tensors of eq. (12) can be written as

$$Q(\xi) = \int_{\mathcal{B}} d^{n-1}\varphi \sqrt{\sigma} T_{ij} n^i \xi^j, \quad (13)$$

where  $\sigma$  is the determinant of the metric  $\sigma_{ab}$ , and  $\xi^i$  and  $n^j$  are the Killing vector field and the unit normal vector on the boundary  $\mathcal{B}$ . For boundaries with timelike ( $\xi = \partial_t$ ) and rotational ( $\xi = \partial_{\varphi}$ ) Killing vector fields, one obtains the quasilocal mass and angular momentum

$$M = \int_{\mathcal{B}} d^{n-1}\varphi \sqrt{\sigma} T_{ij} n^i \xi^j \quad (14)$$

$$J = \int_{\mathcal{B}} d^{n-1}\varphi \sqrt{\sigma} T_{ij} n^i \xi^j \quad (15)$$

provided the surface  $\mathcal{B}$  contains the orbits of  $\xi$ . These quantities are, respectively, the conserved mass and angular momentum of the system enclosed by the boundary  $\mathcal{B}$ . Note that they will both be dependent on the location of the boundary  $\mathcal{B}$  in the spacetime, although each is independent of the particular choice of foliation  $\mathcal{B}$  within the surface  $\partial\mathcal{M}$ . In the context of AdS/CFT correspondence, the limit in which the boundary goes to infinity is taken, and the counterterm prescription ensures that the action and conserved charges are finite.

However, in the presence of a non-trivial dilaton field, the asymptotic behavior of the spacetime is neither dS ( $\Lambda > 0$ ) nor AdS ( $\Lambda < 0$ ). In fact, it has been shown that with the exception of a pure cosmological constant potential, where  $\zeta = 0$ , no asymptotically AdS or dS static spherically symmetric solution exist [25]. But, as in the case of asymptotically AdS spacetimes, according to the domain-wall/QFT correspondence [24], there may be a suitable counterterm for the stress energy tensor which removes the divergences. For the spacetimes with zero curvature boundary [ $R_{ijkl}(\gamma) = 0$ ], the counterterm (7) for the action is proportional to  $\gamma_{ij}$ . Thus, the finite stress-energy tensor in  $(n+1)$ -dimensional Einstein-dilaton gravity with cosmological constant may be written as

$$T^{ij} = \frac{1}{8\pi} \left\{ (K^{ij} - K\gamma^{ij}) - \frac{n-1}{l_{\text{eff}}} \gamma^{ij} \right\}, \quad (16)$$

where  $l_{\text{eff}}$  is given by

$$l_{\text{eff}}^2 = \frac{(n-1)\zeta^2 - n(n-1)}{2\Lambda} e^{-\zeta\Phi}. \quad (17)$$

As  $\zeta$  goes to zero, the effective  $l_{\text{eff}}^2$  of eq. (17) reduces to  $l^2 = -n(n-1)/2\Lambda$  of the AdS spacetimes. The first two terms in eq. (16) is the variation of the action (5) with respect to  $\gamma_{ij}$ , and the last term is the counterterm which removes the divergences. One may note that the counterterm has the same form as in the case of asymptotically AdS solutions with zero curvature boundary, where  $l$  is replaced by  $l_{\text{eff}}$ . Again the conserved quantities can be obtained through the use of eq. (13) with the stress tensor given by eq. (16).

#### 4. Thermodynamics of $(n+1)$ -dimensional rotating black branes with first order string corrections

Here I consider the thermodynamics of solutions of the field equations (3) and (4) with first order string corrections ( $\alpha = 0$ ). The rotation group in  $(n+1)$ -dimensions is  $SO(n)$  and therefore the number of independent rotation parameters for a localized object is equal to the number of Casimir operators, which is  $[n/2] = k$ , where  $[n/2]$  is the integer part of  $n/2$ . It is a matter of calculation to show that the following solution satisfies the field equations (3) and (4) with  $\alpha = 0$ :

$$ds^2 = -f(r) \left( \Xi dt - \sum_{i=1}^k a_i d\phi_i \right)^2 + \frac{r^2}{l^4} e^{\zeta\Phi} \sum_{i=1}^k \left( a_i dt - \Xi l^2 d\phi_i \right)^2 + \frac{r^2}{l^2} e^{\zeta\Phi} \sum_{i=1}^k \left( a_i d\phi_j - a_j d\phi_i \right)^2 + \frac{dr^2}{f(r)} + \frac{r^2}{l^2} \sum_{i=1}^{n-k} (dx^i)^2, \quad (18)$$

$$\Xi^2 = 1 + \sum_{i=1}^k \frac{a_i^2}{l^2},$$

where  $a_i$ 's are the  $k$  rotation parameters,  $dX^2$  is the Euclidean metric on the  $(n-k-1)$ -dimensional submanifold with volume  $V_{n-k-1}$ , and  $\Phi(r)$  and  $f(r)$  are

$$\Phi(r) = \frac{n-1}{4} \ln\left(\frac{c}{r}\right),$$

$$f(r) = \frac{-8\Lambda c}{(n-1)^2} r - \frac{m}{r^{(n-3)/2}},$$

in which  $c$  is an arbitrary constant. It is worthwhile to note that the Ricci scalar goes to zero as  $r \rightarrow \infty$ , and becomes infinite as  $r$  goes to zero.

The Kretschmann scalar diverges at  $r = 0$ , it is finite

for  $r \neq 0$ , and goes to zero as

$\rightarrow \infty$ . Thus, there is an essential singularity located at  $r = 0$ . The spacetime is neither asymptotically flat nor AdS, but has a regular event horizon for negative  $\Lambda$  at:

$$r_h = \left\{ \frac{(n-1)^2 m}{-8\Lambda c} \right\}^{2/(n-3)}. \quad (19)$$

One can obtain the temperature and angular velocity of the horizon by analytic continuation of the metric. The analytical continuation of the Lorentzian metric by  $t \rightarrow i\tau$  and  $a_i \rightarrow ia_i$  yields the Euclidean section, whose regularity at  $r = r_h$  requires that one should identify  $\tau \approx \tau + \beta_h$  and  $\varphi_i \approx \varphi_i + i\beta_h \Omega_i$ , where  $\beta_h$  and  $\Omega_i$  are the inverse Hawking temperature and the angular velocities of the horizon. It is a matter of calculation to show that

$$T_h = \frac{f'(r_h)}{4\pi\Xi} = \frac{(n-1)^2 m}{\pi\Xi} r_h^{(n-3)/2}, \quad (20)$$

$$\Omega_i = \frac{a_i}{\Xi l^2}. \quad (21)$$

Since the area law of entropy is universal, and applies to all kinds of black holes and black branes in Einstein gravity, the entropy per unit volume  $V_{n-k-1}$  of the black brane is

$$S = (2\pi)^k \frac{\Xi (cr_h)^{(n-1)/2}}{4l^{n-k-1}}, \quad (22)$$

where  $r_h$  is the horizon radius.

The mass and angular momentum per unit volume  $V_{n-k-1}$  of the black brane when the boundary  $B$  goes to infinity can be calculated through the use of eqs. (14)-(17),

$$M = (2\pi)^k \frac{c^{(n-1)/2}}{32\pi^{n-k-1}} (n-1)\Xi^2 m, \quad (23)$$

$$J_i = (2\pi)^k \frac{c^{(n-1)/2}}{32\pi^{n-k-1}} (n-1)\Xi m a_i. \quad (24)$$

I now obtain the mass as a function of the extensive quantities  $S$  and  $J_i$ 's. Using the expression for the entropy, the mass, and the angular momenta, given in eqs. (22)-(24), and the fact that  $f(r_h) = 0$ , one can obtain a Smarr-type formula as

$$M(S, \mathbf{J}, Q) = \frac{\sqrt{\Lambda^2 c^2 l^2 S^2 + (n-1)^2 \pi^2 \sum_{i=1}^k J_i^2}}{(n-1)l} \quad (25)$$

One may then regard the parameters  $S$  and  $J_i$ 's as a complete set of extensive parameters for the mass  $M(S, \mathbf{J})$  and define the intensive parameters conjugate to  $S$  and  $J_i$ . These quantities are the temperature and the angular velocities

$$T = \left( \frac{\partial M}{\partial S} \right)_{J, Q}, \quad \Omega_i = \left( \frac{\partial M}{\partial J_i} \right)_{S, Q}. \quad (26)$$

It is a matter of straightforward calculation to show that the intensive quantities calculated by eq. (26) coincide with temperature and angular velocities found previously in eqs. (20) and (21). Thus, the thermodynamic and conserved quantities calculated above satisfy the first law of thermodynamics,

$$dM = Tds + \sum_{i=1}^k \Omega_i dJ_i. \quad (27)$$

## 5. Stability in the canonical ensemble

The stability of a thermodynamic system with respect to the small variations of the thermodynamic coordinates is usually performed by analyzing the behavior of the entropy  $M(S, \mathbf{J})$  around the equilibrium. The local stability in any ensemble requires that  $M(S, \mathbf{J})$  be a convex function of their extensive variables. Thus, the local stability can in principle be carried out by finding the determinant of the Hessian matrix of  $S$  with respect to its extensive variables  $X_i$ ,  $\mathbf{H}_{X_i X_j}^S = \left| \partial^2 S / \partial X_i \partial X_j \right|$  [27]. Also, one can perform the stability analysis through the use of the Hessian matrix of the mass with respect to its extensive parameters [28]. In our case the entropy  $S$  is a function of the mass, and the angular momenta. The number of thermodynamic variables depends on the ensemble which is used. In the canonical ensemble, the angular momenta are fixed parameters, and therefore the positivity of the thermal capacity  $C_J = T(\partial S / \partial T)_J$  is sufficient to assure the local stability. The thermal capacity  $C_J$  per unit volume at constant angular momenta is

$$C_J = \frac{\Xi^3 (cr_h)^{(n-1)/2}}{4(\Xi^2 - 1)l^{n-k-1}} \quad (28)$$

which is positive. Thus, the  $(n+1)$ -dimensional rotating black brane solution (18) is locally stable in the canonical ensemble.

## 6. Closing remarks

In this paper I gave a brief review of the field equations of classical gravity obtained by the variation of the string tree level effective action for the massless boson sector with Lovelock term and also reviewed the counterterm method inspired by the AdS/CFT correspondence for computing the conserved quantities of asymptotically (A)dS spacetimes. In zero order in  $\alpha$ , I found the rotating black brane solution of the field equations with zero curvature horizon, and computed its thermodynamic quantities such as temperature and entropy. Although the solutions of these equations are not asymptotically AdS, I found that one can use the counterterm method for the solutions with zero curvature boundary in order to calculate the conserved quantities. I used this method and obtained the mass and angular momentum of the rotating solutions and found that they satisfy the first law of thermodynamics.

Also, I studied the phase behavior of the rotating black branes in  $(n+1)$  dimensions and showed that there is no Hawking-Page phase transition in spite of the angular momentum of the branes. Indeed, I calculated the heat capacity and found that it is positive for all the phase space, which means that the brane is stable for all the allowed values of the metric parameters discussed in

Sec. IV. This phase behavior is incommensurable with the fact that there is no Hawking-Page transition for black object whose horizon is diffeomorphic to  $\mathbb{R}^p$  and therefore the system is always in the high temperature phase [7].

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