Collapsing spherical null shells in general relativity

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Abstract
In this work, the gravitational collapse of a spherically symmetric null shell with the flat interior and a charged Vaidya exterior spacetimes is studied. There is no gravitational impulsive wave present on the null hypersurface which is shear-free and contracting. It follows that there is a critical radius at which the shell bounces and starts expanding.

Keywords: general relativity, null shell, gravitational collapse

1. Introduction
Many practical problems of general relativity and cosmology involve idealized models constructed by gluing two regions with different metrics across a hypersurface or thin shell having a $\delta$-function singularity in its Riemann tensor due to the discontinuity in the metric's transverse derivative across the shell. The description of timelike or spacelike (non-null) thin shells is well known within general relativity since the outstanding work of Israel [1]. Later, an extension of the Israel formalism to the null or lightlike case was presented by Barrabes and Israel [2]. Then, Poisson introduced a user-friendly reformulation of the Barrabes-Israel original work together with an illustration of the formalism [3] (see also [4] and references therein). Fundamental difference between the null and the non-null case consists in the fact that there are no non-null shells without matter. On the other hand, gravitational “shock waves”, i.e. null shells without any matter, are perfectly allowed by the theory. Nevertheless, general relativity admits also null-like matter, which couples consistently to gravity.

Spherically symmetric null shells have been used for modeling a number of cases, such as black hole interior [5] and thin shell wormholes[6]. They have also been utilized to provide classical models for quantum processes relevant to some aspects of black hole physics such as Hawking radiation and the entropy of a black hole [5, 7]. Another interesting application of null shells concerns the interaction of matter and gravitational field. This problem, which is in general difficult to tackle for arbitrary sources, can be simplified by reducing it to the collision between two null shells. It can be solved analytically with the assumption of spherical symmetry [2].

In this paper, we study the dynamics of a collapsing spherical null shell in flat spacetime, with the charged Vaidya as the exterior. For this purpose, we use Barrabes-Israel null shell formalism [2] to investigate the matching and find the junction conditions. Section 2 is devoted to the formulation of the problem and the matching conditions. In section 3 we investigate the cases of gravitational wave and Bianchi identities. The conclusion follows in section 4.

Natural geometrized units, in which $G = c = 1$ are used throughout the paper. The null hypersurface is denoted by $\Sigma$. The symbol $|_{\Sigma}$ means "evaluated on the null hypersurface ". Latin indices range over the intrinsic coordinates of $\Sigma$ denoted by $\xi^a$ and Greek indices over the coordinates of the 4 manifolds. As we are going to work with distributional valued tensors, there may be terms in a tensor quantity $F$ proportional to some $\delta$ function. These terms are indicated by $\delta^F$.

2. Matching conditions
Consider the gravitational collapse of a thin spherical shell. We imagine that the collapse proceeds at the speed of light in such a manner that the history of the shell coincides with a null hypersurface $\Sigma$. Taking spacetime to be flat inside the shell ($M_-$), we write the metric there as

$$ds^2 = -dt^2 + dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \tag{1}$$
For the exterior space time of the shell \((M_+)\), we then take an incoming charged Vaidya geometry described by the Reissner-Nordstrom-Vaidya metric in the \((v,r,\theta,\phi)\) coordinates
\[
ds_+^2 = \left(1 - \frac{2M(v)}{r} + \frac{Q^2}{r^2}\right) dv^2 + 2 dvdr + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right),
\]
where \(v\) represents advanced Eddington time, at which \(r\) is decreasing towards the future along a ray \(v = \text{const.}\), \(M(v) > 0\) is an arbitrary function of the ingoing null coordinate \(v\), representing the mass accreted at time \(v\), and \(Q\) is the electric charge. The energy-momentum tensor corresponding to the metric (2) has the following form [6],
\[
T_+^{\mu\nu} = T_{\text{em}}^{\mu\nu} + T_{\text{rad}}^{\mu\nu},
\]
and the radiation part is
\[
T_{\text{rad}}^{\mu\nu} = \frac{1}{4\pi r^2} \frac{dM(v)}{dv} k^\mu k^\nu,
\]
with the null vector \(k_\alpha = -\partial_\alpha v\). To glue the interior Minkowski region to the exterior charged Vaidya spacetime along the null hypersurface \(\Sigma\), we need to have
\[
r = r(t) \quad \frac{dr}{dt} |_\Sigma = -1,
\]
as viewed from \(M_-\), while the hypersurfaces \(v = \text{const.}\) turn out to be null, as viewed from \(M_+\). Taking \(z^\mu = (\lambda, \theta, \phi)\), while identifying \(-r\) with the parameter \(\lambda\) on the null generators of the hypersurface, we calculate the tangent basis vectors \(e_\alpha = \partial_\alpha / \partial z^\alpha\) on both sides of \(\Sigma\). Using eq. (6) we get
\[
e^\mu_\phi |_\Sigma = \delta^\mu_\phi, \quad e^\mu_\theta |_\Sigma = (1, -1, 0, 0), \quad e^\mu_\lambda |_\Sigma = (0, 0, 0, 1).
\]
Choosing the tangent-normal vector \(n^\mu\) to coincide with the tangent basis vector associated with the parameter \(\lambda\), so that \(n^\mu = e^\mu_\lambda\), we make sure of generating the null hypersurface \(\Sigma\) by the geodesic integral curves of the future pointing null vector field \(\partial / \partial \lambda\). We may then complete the basis by a transverse null vector \(N^\mu\) uniquely defined by the four conditions
\[
n_\mu N^\mu = -1, \quad n_\mu e^\mu_\lambda = 0 \quad (A = \theta, \phi), \quad \text{and} \quad N^\mu N^\mu = 0 \quad [3].
\]
We find
\[
N^\mu |_\Sigma = \frac{1}{2} (-1, 1, 0, 0),
\]
and the induced metric on \(\Sigma\) given by \(g_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b |_\Sigma\) is computed to be
\[
g_{ab} = \text{diag}(0, r^2, r^2 \sin^2 \theta) |_\Sigma,
\]
which is the same on both sides of the hypersurface. Defining a pseudo-inverse of the induced metric \(g_{ab}\) on \(\Sigma\) as \(g^{ac} g_{bc} = \delta^a_b + n^a N^b \delta^c_d\), with \(n^a = \delta^a_\lambda\) [2], one gets
\[
g^{ab} = \text{diag} \left(0, r^{-2}, \frac{1}{r^2 \sin^2 \theta}\right) |_\Sigma.
\]
The final junction conditions are formulated in terms of the jump in the extrinsic curvature. Using the definition \(K_{ab} = e^c_a e^b_d \nabla_c N_d\) [2], we compute the transverse extrinsic curvature tensor on both sides of \(\Sigma\). Its non-vanishing components on the minus side are found as
\[
K_{00} |_{-} = \sin^2 \theta K_{\phi\phi} |_{-} = -\frac{r}{2} |_\Sigma.
\]
The corresponding non-vanishing components on the plus side are
\[
K_{00} |_{+} = \sin^2 \theta K_{\phi\phi} |_{+} = \frac{r}{2} \left(1 - \frac{2M(v)}{r} + \frac{Q^2}{r^2}\right) |_\Sigma.
\]
The jump in the transverse extrinsic curvature across the null hypersurface given by \(\gamma_{ab} = 2[K_{ab}]\) has the following non-vanishing components
\[
\gamma_{00} = \sin^2 \theta \gamma_{\phi\phi} = 2 \left(\frac{Q^2}{r^2} - 2M(v)\right) |_\Sigma.
\]
Computing the acceleration parameter \(\kappa\) by \(n^\mu \nabla_\mu n^\nu = \kappa n^\nu\) on both sides of the hypersurface, we get
\[
\kappa_+ = \kappa_- |_{\Sigma} = 0
\]
indicating that \(\lambda\) is an affine parameter on each side of the hypersurface [3]. The surface energy-momentum tensor of the lightlike shell having the null hypersurface \(\Sigma\) as its history is directly related to the jump in the transverse extrinsic curvature. In the tangent basis \(e^\alpha_a\), it can be written in the form [2]
\[
S^{\alpha\beta} = \int n^a f + n^a g^{ab} \gamma_{ab} |_\Sigma,
\]
where \(f\) and \(p\) are the surface energy density and isotropic surface pressure of the shell, respectively, measured by a family of freely-moving observers crossing the null hypersurface. Using the jumps in the extrinsic curvature obtained above, we calculate them as
\[
16\pi f = -g^{ab} \gamma_{ab}
\]
\[
= \frac{2}{r^2} K_{00} |_\Sigma
\]
\[
= \frac{2}{r^2} \left(2M - \frac{Q^2}{r}\right) |_\Sigma,
\]
Furthermore, the induced metric on \(\Sigma\) given by
\[
g_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b |_\Sigma
\]
is decreased towards the future along a ray.

\[
N_\mu |_\Sigma = \frac{1}{2} (-1, 1, 0, 0),
\]

\[
N_\mu |_\Sigma = \frac{1}{2} \left(1 - \frac{2M(v)}{r} + \frac{Q^2}{r^2}\right),
\]

\[
|_\Sigma = \frac{1}{2} \left(1 - \frac{2M(v)}{r} + \frac{Q^2}{r^2}\right).
\]
\[16 \pi p = - \gamma_{ab} n^a n^b \]
\[= - \gamma_{ab} \]
\[= 0. \]

Therefore, the shell is only admitting a surface energy density given by eq. (16). In addition, from (16) we see that this energy density is positive, provided \( r > \frac{Q^2}{2M} \) from the point of \( r_0 = \frac{Q^2}{2M}, \) where \( r_0 = M - \sqrt{M^2 - Q^2} \) is the Cauchy horizon, to \( r = 0, \) the sign of \( f \) turns out to be negative that is forbidden from the physical point of view. Hence, as Dray has suggested [8] it seems more natural to expect that the shell should bounce at the radius \( r_0 \) in order to avoid having a negative energy density.

### 3. Gravitational wave and Bianchi identities

The presence of gravitational waves having the null hypersurface \( \Sigma \) as history is seen in the following way. Let us first construct a null tetrad frame on \( \Sigma. \) Consider a congruence of timelike geodesics with continuous 4-velocity \( u \) across \( \Sigma \) so that \([u,u] = [\mu \mu] = 0. \) On the null hypersurface \( \Sigma, \) we have the normal \( n^\alpha, \) which is tangential to \( \Sigma, \) and the timelike vector field \( u^\alpha \) crossing the null hypersurface such that \( u^\alpha n_\alpha = -s < 0. \) It is then advantageous to introduce on \( \Sigma \) a transverse null vector field \( l^\mu \) defined by \( l^\mu = -\frac{1}{2s^2} n^\mu + \frac{1}{s} u^\mu, \) satisfying the normalization condition \( l^\mu n_\alpha = -1 \) [9]. Let next \( m^\mu \) and \( \tilde{m}^\mu \) be a complex covariant vector field and its complex conjugate (indicated by a bar) being chosen so that they are null \( (m^\alpha m_\alpha = \tilde{m}^\alpha \tilde{m}_\alpha = 0), \) tangent to \( \Sigma, \) orthogonal to \( n^\mu \) and \( l^\mu, \) and satisfy \( m^\alpha \tilde{m}_\alpha = 1. \) Now \( n^\mu, l^\mu, m^\mu, \) and \( \tilde{m}^\mu \) constitute the desired null tetrad frame on \( \Sigma \) which will be used in the following. We get
\[l^\mu = (1,1/2,0,0), \]
\[m^\mu = \frac{1}{r \sqrt{2}} \left(0, 0, i, \frac{1}{\sin \theta} \right) \xi, \]
\[\tilde{m}^\mu = \frac{1}{r \sqrt{2}} \left(0, 0, -i, \frac{1}{\sin \theta} \right) \xi. \]

Using this null tetrad, the Newman-Penrose component of the singular part of the Weyl tensor of Petrov type \( N \) characterizing an impulsive gravitational wave with history \( \Sigma, \) is calculated as [9]
\[\bar{\psi}_4 = \frac{1}{2} \gamma_{ab} m^a m^b \]
\[= \frac{1}{2} \left( \gamma_{\phi \phi} (\tilde{m}^a)^2 + \gamma_{\phi \phi} (m^a)^2 \right) = 0. \]

Where we have used eqs. (13) and (20). This shows explicitly that there is no impulsive gravitational wave present and the null hypersurface \( \Sigma \) is just the history of a lightlike shell of matter being characterized by the surface energy density given by eq. (16). In this case, induced geometry on \( \Sigma \) inherited from the embedding spacetimes, is of type \( III \) according to a classification introduced by Penrose[10].

The expansion \( \theta \) and complex shear \( \sigma \) of the geodesic generators of the null hypersurface \( \Sigma \) can now be defined by the following relations using the null tetrad [9].
\[\theta = m^\mu \tilde{m}^\nu \nabla_\nu n_\mu = -\frac{1}{r} \xi, \]
\[\sigma = m^\mu m^\nu \nabla_\nu n_\mu = 0. \]

This shows that \( \Sigma \) is a future null cone generated by the shear-free null geodesics with the expansion as given by eq. (22).

The intrinsic properties of the pressureless null shell embodied in \( f, \theta, \sigma, \) and \( \psi_B (\theta = 1, 2, 3, 4), \) are related to the outside medium described by \( T^{\mu \nu} \) and the outside geometry described by \( \psi_B \), through the following Bianchi identities [9].

\[\left[ T_{\mu \nu} n^\mu n^\nu \right] = 0, \]
\[\left[ \psi_2 \right] - \frac{2\pi}{3} [T] = \sigma \psi_4 - 4\pi \left( f_{,\mu} n^\mu + (\theta + \kappa) f \right) \]
\[- \left[ T_{\mu \nu} l^\mu n^\nu \right] = f_{,\mu} n^\mu + (2\theta + \kappa) f, \]

where \( \left[ \psi_2 \right] \) is the jump in the Newman-Penrose component of the Weyl tensors of \( M_- \) and \( M_+ \) across \( \Sigma. \) On the null tetrad constructed above, \( \psi_2 \) is given by [4].
\[\psi_2 = C_{\mu \nu \rho \sigma} n^\mu n^\nu \tilde{m}^\rho l^\sigma, \]

where in our case, the jump \( \left[ \psi_2 \right] \) is computed as
\[\left[ \psi_2 \right] = \frac{Q^2 - Mr}{r^4} \xi. \]

The vanishing of the jump \( \left[ T_{\mu \nu} n^\mu n^\nu \right] \) in eq. (24) can easily be seen by using eqs. (3), (4), (5), (7), and noting that \( k_{,\mu} n^\mu = 0. \) It means that the shell does not interact with the outside medium. Note that the energy-momentum tensor corresponding to the metric (2) given by the expression (3), is traceless (so \( [T] = 0), \) and from eq. (14) for a pressureless null shell, we found \( [\kappa] = 0, \) and the null hypersurface \( \Sigma \) is generated by shear-free \( (\sigma = 0) \) null generators, so that from (16) and (22), we may calculate the right hand side of eq. (25) as
\[-4\pi (f_{,\mu} n^\mu + (\theta + \kappa) f) = \frac{Q^2 - Mr}{r^4} \xi. \]

Hence by the virtue of (28), we can see that eq. (25) holds in our case. Concerning eq. (26), we use the above equations to get
\[ f,\mu \eta + (2\theta + \kappa) f = -\frac{Q^2}{8\pi T^4} \downarrow \Sigma, \quad \text{and} \quad [T_{\mu \nu} \Upsilon^\mu \Upsilon^\nu] = -\frac{Q^2}{8\pi T^4} \downarrow \Sigma, \]

and eq. (26) is also satisfied identically.

4. Conclusion

Applying the Barrabes-Israel null shell formalism, we have examined the gravitational collapse of a spherical null shell with the flat Minkowski interior and the charged Vaidya exterior geometry. The history of \( \Sigma \) of the shell, in this case, is a future null cone which is a shearless and contracting null hypersurface. We see that there is no gravitational wave present \( \psi_4 = 0 \) and the lightlike matter shell is simply characterized by a surface energy density given by eq. (16) with no surface pressure. We have also demonstrated that there is a critical radius within the Cauchy horizon at which the collapsing shell should bounce and start expanding, thereby avoiding unphysical negative energy densities.

References