Thermal gravitational waves in accelerating universe

B Ghayour
School of Physics, University of Hyderabad, Hyderabad 500046, India.

(Received 24 February 2013, in final from 15 May 2013)

Abstract
Gravitational waves are considered in thermal vacuum state. The amplitude and spectral energy density of gravitational waves are found enhanced in thermal vacuum state compared to its zero temperature counterpart. Therefore, the allowed amount of enhancement depends on the upper bound of WMAP-5 and WMAP-7 for the amplitude and spectral energy density of gravitational waves. The enhancement of amplitude and spectral energy density of the waves in thermal vacuum state is consistent with current accelerating phase of the universe. The enhancement feature of amplitude and spectral energy density of the waves is independent of the expansion model of the universe and hence the thermal effect accounts for it. Therefore, existence of thermal gravitational waves is not ruled out.

Keywords: gravitational waves; thermal vacuum state

1. Introduction
The inflationary scenario [1] predicts a stochastic cosmic background of gravitational waves (CGWB) [2]. The spectrum of these relic gravitational waves depends not only on the details of the early stage of inflationary expansion but also on the expansion behavior of the subsequent stages, including the current epoch of the universe. Computation of the gravitational waves spectrum for matter dominated stage of the universe is usually done in a decelerated expansion model [3-8]. The resulting spectrum is used for putting constrains on the observations of the gravitational waves that are originate from sources other than the early universe. The astronomical observations of SN Ia [9, 10] indicate that the universe is currently under accelerating expansion with non zero cosmological constant. The observed acceleration of the present universe is supposed to be driven by the dark energy according to the ΛCDM concordance model. The effect of current acceleration of the universe on the spectrum and spectral energy density of the relic gravitational waves has been studied [4, 11]. It is noticed that the shape, amplitude and spectrum of the relic gravitational waves change due to the current accelerating phase of the universe [11].

The existence of thermal graviton background with the black body type spectrum has received much attention recently [2, 12]. If the inflation was preceded by a radiation era, then there would be a thermal background of gravitational waves at the time of inflation [12]. Also, the generation of tensor perturbation during inflation by the stimulated emission process leads to the existence of thermal background of gravitational waves [13]. The direct detection of the thermal gravitons is challenging but may be possible in near future with the 21-cm emission line of atomic hydrogen. The other but indirect way to see the existence of thermal gravitons is via the cosmic microwave background radiation (CMB). This is due to the fact that the gravitational wave background could leave an observable imprint on the temperature and polarization anisotropies of the CMB [14]. It is shown that the B mode angular power spectrum of CMB get enhanced due to thermal gravitons but it is less than the upper bound of Wilkinson Microwave Anisotropy Probe (WMAP ) 3-year data [12]. Though the B mode polarization is not detected directly (may be with the Planck mission [15]), the enhancement of the angular power spectrum is to be addressed for the compatibility with current data. For example, the recent WMAP 7-year result gives an upper bound on the B mode power spectrum, \( \frac{l(l+1)}{2\pi}C_l^{BB} < 0.55(\mu_K)^2 \) [16]. And the other observations are giving even higher upper bounds on the power spectrum, (see for example, DASI [17], Boomerang [18], Maxipol [19], QUaD [20], CBI [21], Capmap [22]). These results show that there exists a
discrepancy between the standard theoretical and observed power spectra of the B mode polarization. This issue is also motivating for investigating the effect of thermal gravitons on the various angular power spectra of CMB anisotropy. However, we studied the impact of the decelerated and accelerated expansion of the universe on the spectrum and spectral energy density of gravitational waves in thermal vacuum state in [23]. The results showed the enhancement of the spectrum and spectral energy density was due to $T = 0.001 \text{Mpc}^{-1}$. But in present work, we compare the amount of enhancement due to the temperatures $T = 0.001 \text{Mpc}^{-1}$, $T = 0.01 \text{Mpc}^{-1}$ with the upper bounds of WMAP-5 and WMAP-7 for the spectrum and spectral energy density of gravitational waves. Therefore, by this comparison, we can investigate the allowed amount of enhancement corresponding to the upper bounds of WMAP.

The obtained spectrum is normalized with the WMAP spectrum and used to derive the bounds on the observation of the gravitational waves generated from sources other than early universe. The peak position of the spectrum at lower side of the thermal gravitational waves is unaltered in the accelerated universe but gets enhanced in comparison with its zero temperature case. It is shown that the spectral energy density of the gravitational waves in thermal vacuum state does not exceed the bound put by primordial nucleosynthesis rate. In the present work, we use the unit $c = h = k_B = 1$.

2. Gravitational waves spectrum in thermal state

The perturbed metric for a homogeneous isotropic flat Friedmann-Robertson-Walker (FRW) universe can be written as

$$ds^2 = S^2(\eta)\left(d\eta^2 - (\delta_{ij} + h_{ij})dx^i dx^j\right),$$

(2)

where $S(\eta)$ is the cosmological scale factor, $\eta$ is the conformal time and $\delta_{ij}$ is the Kronecker delta symbol. The $h_{ij}$ are metric perturbations field containing only the pure gravitational waves and is transverse-traceless i.e.;

$$\nabla_i h^{ij} = 0, \delta^{ij} h_{ij} = 0.$$

The present study mainly deals with amplitude and spectral energy density of the relic gravitational waves that are generated by the expanding space-time of background. Hence, the perturbed matter source is not taken into account in the present work. Since the relic gravitational waves are very weak, one needs to consider only the linearized field equation given by

$$\nabla_i \left(\sqrt{-g} \nabla^i h_{ij}(x, \eta)\right) = 0.$$

(2)

The tensor perturbations have two independent degrees of freedom, which can be chosen as $h^+$ and $h^\times$ polarization modes. To compute the spectrum of gravitational waves $h(x, \eta)$, we express $h^+$ and $h^\times$ in terms of the creation ($a^\dagger$) and annihilation ($a$) operators,

$$h_{ij}(x, \eta) = \sqrt{16\pi l_{pl}^2} \int_p \frac{d^3k}{(2\pi)^3/2} a_p^\dagger k_{ij}(k) S(\eta)$$

$$\times \frac{1}{\sqrt{2k}} \left[ a_p^{ij} k_p^{ij}(\eta) e^{ik \cdot x} + a_p^{ij} k_p^{ij}(\eta) e^{-ik \cdot x}\right],$$

(3)

where $k$ is the comoving wave number, $k = |k| l_{pl} = \sqrt{G}$ is the Planck's length and $p = +, \times$ is polarization mode. The polarization tensor $\varepsilon^p_{ij}(k)$ is symmetric and transverse-traceless $k^i k^j (\varepsilon^p_{ij} k) = 0, \delta^i_{ij} e^p_{ij} (k) = 0$, and satisfies the conditions $\varepsilon^p_{ij}(k) \varepsilon^{p\dagger}_{ij}(k) = 2 \delta_{pp}$ and $\varepsilon^p_{ij}(-k) = \varepsilon^p_{ij}(k)$, also $a^\dagger$ and $a$ operators satisfy $\left[a_p^{ij} a^\dagger k_p^{ij}(k)\right] = \delta_{pp} \varepsilon^3(k - k^\dagger)$ and the initial vacuum state is defined as:

$$a_p^{ij}(0|0) = 0,$$

(4)

for each $k$ and $p$. The energy density of the gravitational waves in vacuum state is

$$t_{00} = \frac{1}{32\pi l_{pl}^2} h_{ij} h^{ij}_{00}.$$  

For a fixed number, $k$, and a fixed polarization $p$ the linearized wave eq. (2) gives

$$h_{ij} + 2 \frac{S}{S'} h_{ij} + k^2 h_{ij} = 0,$$

(5)

where prime means derivative with respect to the conformal time. Since each polarization state is the same, we denote $h_k(\eta)$ without the polarization index from here onwards.

Next, we rescale the field $h_{k}(\eta)$ by taking $h_{k}(\eta) = f_k(\eta)/S(\eta)$, where the mode functions $f_k(\eta)$ obey the minimally coupled Klein-Gordon equation

$$f''_{k} + \left[k^2 - \frac{S''}{S}\right]f_k = 0.$$

(6)

The general solution of the above equation is a linear combination of Hankel's function with a generic power-law for the scale factor $S = \eta^n$ given by

$$f_k(\eta) = A_k \sqrt{\eta} H^{(1)}_{\nu}(k \eta) + B_k \sqrt{\eta} H^{(2)}_{\nu}(k \eta).$$

(7)

For given a model of the expansion of universe, consisting of a sequence of successive scale factor with different $n$'s, we can work out an exact solution $f_k(\eta)$ by matching its values and its derivatives at the joining points.

The approximate computation of the spectrum of the gravitational waves is usually taken in two limiting cases depending on whether the waves are within or outside of the barrier. In the case of the outside barrier
\( k^2 > S'/S \), short wave approximation) the gravitational field leads to decreasing amplitude \( h_k \propto 1/S(\eta) \) while for the waves inside the barrier \( k^2 < S'/S \), long wave approximation), \( h_k = C_k \) is simply a constant. Thus, these results can be used to estimate the spectrum for the present epoch of the universe.

The history of the overall expansion of the universe can be modelled as the following sequence of successive epochs of power-law expansion. The initial stage (inflationary) is:

\[
S(\eta) = l_0 |1 + e^{\beta} - \eta|, \quad -\infty < \eta \leq \eta_1.
\]

Where \( 1 + \beta < 0 \), \( \eta < 0 \) and \( l_0 \) is constant.

The radiation-dominated stage:

\[
S(\eta) = S_{S}(\eta - \eta_{p}), \quad \eta_{p} \leq \eta \leq \eta_{E}.
\]

The matter-dominated stage:

\[
S(\eta) = S_{m}(\eta - \eta_{m})^{2}, \quad \eta_{m} \leq \eta \leq \eta_{E}.
\]

Where \( \eta_{E} \) is the time when the dark energy density \( \rho_{\Lambda} \) is equal to the matter energy density \( \rho_{m} \). Before the discovery of the accelerating expansion of the universe, the current expansion is used to take as decelerating one because of the matter-dominated stage. Now, following the matter-dominated stage, it is reasonable to add an epoch of accelerating stage, which is probably driven by either the cosmological constant, or the quintessence, or some other kind of condensate [25]. The value of the redshift \( z_E \) at the time \( \eta_E \) is given by

\[
1 + z_E = \frac{S(\eta_0)}{S(\eta_E)}, \quad \text{where } \eta_0 \text{ is the present time.}
\]

Since \( \rho_{\Lambda} \) is constant and \( \rho_{m}(\eta) \propto S^{-3}(\eta) \), we get

\[
\frac{\rho_{\Lambda}}{\rho_{m}(\eta_E)} = \frac{\rho_{\Lambda}}{\rho_{m}(\eta_0)(1 + z_E)^3} = 1.
\]

If the current value of \( \Omega_{\Lambda} \sim 0.7 \) and \( \Omega_{m} \sim 0.3 \), then it follows that

\[
1 + z_E = \frac{\Omega_{\Lambda}}{\Omega_{m}} \sim 1.33.
\]

The accelerating stage (up to the present) is:

\[
S(\eta) = l_0 |\eta - \eta_a|^{-1}, \quad \eta_E \leq \eta \leq \eta_0.
\]

This stage describes the accelerating expansion of the universe, which gives a new feature and hence can influence the spectrum of relic gravitational waves. It is noted that the actual scale factor function \( S(\eta) \) differs from eq. (14), since the matter component exists in the current universe. However, the dark energy is dominant, so (14) is an approximation to the current expansion behavior.

Given \( S(\eta) \) for the various epochs, the derivative \( S' = dS/d\eta \) and the ratio \( S'/S \) follow immediately. Except for the \( \beta_s \) that is imposed as the model parameter, there are ten constants in the expressions of \( \eta \). By the continuity conditions of \( S(\eta) \) and \( S'(\eta) \) at the four given joining points \( \eta_1, \eta_2, \eta_3, \) and \( \eta_{E} \), one can fix only eight constants. The remaining two constants can be fixed by the overall normalization of \( S \) and by the observed Hubble constants the expansion rate. Specifically, we put \( |\eta - \eta_a| = 1 \) as the normalization of \( S \) which fixes the \( \eta_a \) and the constant \( l_0 \) is fixed by the following calculation:

\[
\frac{1}{H} = \frac{S^2}{S_{E}} |\eta_a| = l_0.
\]

Where \( l_0 \) is the Hubble radius at present.

In the expanding FRW universe, the physical wavelength is related to the comoving wave number by

\[
\lambda = \frac{2 \pi S(\eta)}{k},
\]

and the wave number \( k_0 \) corresponding to the present Hubble radius is

\[
k_0 = \frac{2 \pi S(\eta_0)}{l_0} = 2 \pi.
\]

There is another wave number

\[
k_E = \frac{2 \pi S(\eta_E)}{1 + z_E},
\]

whose corresponding wavelength at the time \( \eta_E \) is the Hubble radius \( 1/H \).

By matching \( S \) and \( S'/S \) at the joint points, we have derived, for example,

\[
l_0 = \frac{S_S(\eta_0)}{S_{E}} \frac{1}{1 - \beta - 2}, \quad \beta = \frac{1}{2} (1 + \beta).
\]

Where \( b = 1 + \beta \), which is defined differently from [26], \( \frac{\beta}{2} \frac{1}{\eta_{S}} \). With these specifications, the functions \( S(\eta) \) and \( S'(\eta)/S(\eta) \) are fully determined. In particular, \( S'(\eta)/S(\eta) \) rises up during the accelerating stage, instead of decreasing as in the matter-dominated stage. This causes modifications to the spectrum of relic gravitational waves.

The power spectrum of gravitational waves is defined
as:
\[ \int h^2(k, \eta) \frac{dk}{k} = \langle 0 \mid h^\dagger(x, \eta) h(x, \eta) \mid 0 \rangle, \]  
(20)
Substituting eq. (3) in (20) and taking the contribution from each polarization is the same, and we get
\[ h(k, \eta) = \frac{4\pi}{\sqrt{\pi} k} \left| h(\eta) \right| \]  
(21)
Thus once the mode function \( h(\eta) \) is known, the spectrum \( h(k, \eta) \) follows.

The spectrum at the present time \( h(k, \eta_0) \) can be obtained, provided the initial spectrum is specified. The initial condition is considered to be the inflationary stage. Thus the initial amplitude of the spectrum is given by
\[ h(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{2+\beta}, \]  
(22)
where \( A = 8\sqrt{\frac{\pi}{\nu k_0}} \) is a constant. The power spectrum for the primordial perturbation of energy density is \( P(k) \propto h(k, \eta_0)^2 \) and in terms of initial spectral index \( n \) is defined as \( P(k) \propto k^{-n} \).

Next, we consider the gravitational waves in the thermal vacuum state and hence estimate its spectrum at the present time. If the graviton field had zero occupation prior to inflation then
\[ [a_k, a_k^\dagger] = \delta^3(k-k'), \]  
(23)
the vacuum satisfies
\[ a_k(0)0 = 0, \]  
(24)
and we would obtain a correlation function \( \sim f_k(\eta)^2 \). But if the field was in thermal equilibrium some earlier epoch it will retain its thermal nature even after decoupling from the rest of the radiation field.

An effective approach to deal with the thermal vacuum state is the thermo-field dynamics (TFD) [27]. In this approach one requires a tilde space besides the usual Hilbert space, and the direct product space is made up of these two spaces. Every operator and state in the Hilbert and tilde space, which satisfy the usual commutation relations, \( [a_k, a_{k'}^\dagger] = [\tilde{a}_k, \tilde{a}_{k'}^\dagger] = \delta^3(k-k') \). And all other commutation relations of these operators are zero. By the appropriate action of the operator (24) on \( a_k \) and \( a_k^\dagger \), we get [28]
\[ \tau^+ a_k = a_k \cosh \delta_k + \tilde{a}_k \sinh \delta_k, \]  
(25)
Hence the occupation number in thermal vacuum state can be written as
\[ \langle a_k^\dagger a_{k'} \rangle = \left( \frac{1}{k} \right) \delta^3(k-k'). \]  
(26)
Therefore the amplitude of the waves in the thermal vacuum state in comparison with eq. (21), can be written [23]:
\[ h(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{2+\beta} \coth^{1/2} \left( \frac{k}{2T} \right), \]  
(27)
The last term becomes significant when the ratio \( k / 2T \) is less than unity. The wave number \( k \) and temperature \( T \) are comoving quantity which are related to the physical parameters at the time of inflation, (for details see [12]). Thus it is expected to have an enhancement of the spectrum by a factor \( \coth^{1/2} \left( \frac{k}{2T} \right) = \coth^{1/2} \left( \frac{H \tau}{2T_i} \right) \).

The amplitude of waves can be considered in different range of wave numbers [11] and we follow the same ranges in the present work for thermal vacuum state of gravitational waves. Thus, the amplitude of the spectrum in different ranges are given by [23]:

a) \( k \leq k_E \)
\[ h_T(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{2+\beta} \coth^{1/2} \left( \frac{k}{2T} \right), \]  
(28)
b) \( k_E < k \leq k_0 \)
\[ h_T(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{\beta-1} \coth^{1/2} \left( \frac{k}{2T} \right) \frac{1}{(1 + \frac{z_E}{k})^3}; \]  
(29)
c) \( k_0 < k \leq k_2 \)
\[ h_T(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{\beta} \coth^{1/2} \left( \frac{k}{2T} \right) \frac{1}{(1 + \frac{z_E}{k})^3}; \]  
(30)
d) \( k_2 < k \leq k_s \)
\[ h_T(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{1+\beta} \left( \frac{k_0}{k_2} \right) \frac{1}{(1 + \frac{z_E}{k})^3}; \]  
(31)
This is the interesting range for the observation point of view of LIGO and LISA detectors.

\[ \tau = \frac{1}{k} \]  
(32)

where \( \omega \) is the resonance frequency of the field. The \( a_k, a_k^\dagger \) and \( \tilde{a}_k, \tilde{a}_k^\dagger \), are respectively the annihilation and creation operators in Hilbert and tilde space, which satisfy the usual commutation relations, \( [a_k, a_{k'}^\dagger] = [\tilde{a}_k, \tilde{a}_{k'}^\dagger] = \delta^3(k-k') \). And all other commutation relations of these operators are zero. By the appropriate action of the operator (24) on \( a_k \) and \( a_k^\dagger \), we get [28]
\[ \tau^+ a_k = a_k \cosh \delta_k + \tilde{a}_k \sinh \delta_k, \]  
(25)
Hence the occupation number in thermal vacuum state can be written as
\[ \langle a_k^\dagger a_{k'} \rangle = \left( \frac{1}{k} \right) \delta^3(k-k'). \]  
(26)
Therefore the amplitude of the waves in the thermal vacuum state in comparison with eq. (21), can be written [23]:
\[ h(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{2+\beta} \coth^{1/2} \left( \frac{k}{2T} \right), \]  
(27)
The last term becomes significant when the ratio \( k / 2T \) is less than unity. The wave number \( k \) and temperature \( T \) are comoving quantity which are related to the physical parameters at the time of inflation, (for details see [12]). Thus it is expected to have an enhancement of the spectrum by a factor \( \coth^{1/2} \left( \frac{k}{2T} \right) = \coth^{1/2} \left( \frac{H \tau}{2T_i} \right) \).

The amplitude of waves can be considered in different range of wave numbers [11] and we follow the same ranges in the present work for thermal vacuum state of gravitational waves. Thus, the amplitude of the spectrum in different ranges are given by [23]:

a) \( k \leq k_E \)
\[ h_T(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{2+\beta} \coth^{1/2} \left( \frac{k}{2T} \right), \]  
(28)
b) \( k_E < k \leq k_0 \)
\[ h_T(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{\beta-1} \coth^{1/2} \left( \frac{k}{2T} \right) \frac{1}{(1 + \frac{z_E}{k})^3}; \]  
(29)
c) \( k_0 < k \leq k_2 \)
\[ h_T(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{\beta} \coth^{1/2} \left( \frac{k}{2T} \right) \frac{1}{(1 + \frac{z_E}{k})^3}; \]  
(30)
d) \( k_2 < k \leq k_s \)
\[ h_T(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{1+\beta} \left( \frac{k_0}{k_2} \right) \frac{1}{(1 + \frac{z_E}{k})^3}; \]  
(31)
This is the interesting range for the observation point of view of LIGO and LISA detectors.
c) \( k_s < k < k_i \)

\[
h(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{1 + \beta_i - \beta_s} \left( \frac{k}{k_0} \right)^{\beta_i} \left( \frac{k_0}{k} \right)^{1 - \beta_i} \left( 1 + z_E \right)^{2 \beta_i}, \tag{32}
\]

This frequency range is beyond the current observations. Note that the temperature dependent factor in the high frequency range is negligible, hence the term is dropped in the expressions (d) and (e).

The overall multiplication factor \( A \) that appears in all the spectra is determined in the absence of the temperature dependent term with the WMAP-3 year data [11]. This is based on the assumption that the contribution from gravitational waves and from the density perturbations are of the same order of magnitude, or if the CMB anisotropies at low multipole are induced by the gravitational waves, it is possible to write \( \Delta T / T \approx h(k, \eta_0) \). The observed CMB anisotropies [29] at lower multipoles is \( \Delta T / T \approx 0.37 \times 10^{-5} \) at \( l \sim 2 \) which corresponds to the largest scale anisotropies that have been observed so far. Thus, taking this to be the perturbations at the Hubble radius gives:

\[
h(k_0, \eta_0) = A \left( \frac{k_0}{k} \right)^{1 + \beta_i - \beta_s} \left( \frac{k_0}{k} \right)^{\beta_i} \left( \frac{k_0}{k} \right)^{1 - \beta_i} \left( 1 + z_E \right)^{2 \beta_i} \tag{33}
\]

However, there is a subtlety in the interpretation of \( \Delta T / T \) at low multipoles, whose corresponding scale is very large \( \sim l_0 \). At present, the Hubble radius is \( l_0 \), and the Hubble diameter is \( 2l_0 \). On the other hand, the smallest characteristic wave number is \( k_E \), whose corresponding physical wave length at present is:

\[
2\pi S(\eta_0) \approx l_0 (1 + z_E) \approx 1.32 l_0, \text{ which is within the Hubble diameter } 2l_0, \text{ and is theoretically observable.}
\]

So, instead of eq. (33), if \( \Delta T / T \approx 0.37 \times 10^{-5} \) at \( l \sim 2 \) is taken as the amplitude of the spectrum at \( v_E \), one would have:

\[
h_T(k_E, \eta_0) = A \left( \frac{k_E}{l_0} \right)^{2 + \beta_i} \left( \frac{k_E}{l_0} \right)^{1 - \beta_i} \left( 1 + z_E \right)^{2 \beta_i}, \text{ yielding a smaller } A \text{ than that in eq. (33) by a factor } (1 + z_E)^{1 - \beta_i} \sim 2.3 \tag{11].}
\]

Next objective is to check the allowed range of \( \beta \). During the inflationary expansion when the \( k \) mode wave enters the barrier with \( \lambda_i = \frac{1}{H(\eta_i)} \), it follows that:

\[
\lambda_i = \frac{l_0}{l} \left( \frac{k_0}{k} \right)^{2 + \beta_i} \text{.}
\]

For the classical treatment of the background gravitational field to be valid, this wavelength should be greater than the Planck length, \( l_{pl} < \lambda_i \); so,

\[
\left( \frac{v}{v_0} \right)^{2 + \beta_i} < \frac{8\sqrt{\pi}}{A}. \tag{34}
\]

At the highest frequency \( v = v_1 \), this gives the following constraint:

\[
\beta < -2 - \frac{\ln \left( \frac{8\sqrt{\pi}}{A} \right)}{\ln \left( \frac{v_1}{v_0} \right)}, \tag{35}
\]

which depends on \( A \). Thus, for \( A \) given in (33), one obtains the upper limit \( \beta < -1.78 \). Putting \( b / l_0 \) given by eq. (19) into \( A \), using \( \frac{v_2}{v_0} = 58.8 \) and \( \frac{l_0}{l_{pl}} = 1.238 \times 10^{61} \), one has:

\[
1.484 \times 10^{58} A \left( \frac{1 + z_E}{3} \right)^3 \approx \left( \frac{v_1}{v_0} \right)^{2 + \beta_i} \left( \frac{v_1}{v_0} \right)^{1 - \beta_i}. \tag{36}
\]

For \( A \) given in (33), and \( \beta = -1.9 \), then \( \beta_s = -0.552 \), (for more details, see [11]).

We obtain the spectrum in the thermal vacuum state with \( k = 2\pi \nu_2, \nu_2 = 1.5 \times 10^{-18} \text{ Hz}, \nu_0 = 2 \times 10^{-18} \text{ Hz}, \nu_1 = 3 \times 10^{10} \text{ Hz}, \nu_2 = 117 \times 10^{18} \text{ Hz}, \nu_s = 10^{18} \text{ Hz} \), (the value \( v_1 \) is taken in such a way that spectral energy density does not exceed the level of \( 10^{-6} \), as required by the rate of the primordial nucleogenesis). The range of frequency is chosen in accordance with generation of gravitational waves that vary from early universe to various astrophysical sources. The spectrum is computed in the thermal vacuum state and the chosen values of the parameters for the accelerated as well as decelerated model with temperature \( T = 0.001 \text{ Mpc}^{-1} \) in [23]. But, we also added \( T = 0.01 \text{ Mpc}^{-1} \) in this work, for comparing the amount of enhancement due to both temperatures with the upper bound of WMAP-5 and WMAP-7. The chosen temperatures are usually considered in the context of thermal gravitons by taking into account the tensor mode of CMB spectrum with WMAP data bound. Since we use the natural unit, the wave number and temperature are computed numerically in the Mpc\(^{-1}\) unit. The amplitude of the spectrum of the thermal gravitational waves is found enhanced compared to its zero temperature case (vacuum case). It is observed that the spectrum for \( T = 0.001 \text{ Mpc}^{-1} \) gets maximum enhancement \( \sim 1.51 \) times more than the vacuum case, at \( l = 2 \) and \( k = k_E \), and it is \( \sim 4.6 \) times for \( T = 0.01 \text{ Mpc}^{-1} \).

The plots for the amplitude of spectrum \( h_T(k, \eta_0) \) versus the frequency \( \nu \) for \( \beta = -1.9 \) and \( \beta_s = -0.552 \) are given in figure 1. The amplitude of the spectrum gets enhanced for the frequency range, \( 10^{-19} \text{ Hz} \leq \nu \leq 1.49 \times 10^{-17} \text{ Hz} \), due to the thermal effect of gravitational waves but not for the frequency range.
1.49×10^{-17} \text{ Hz} \leq \nu \leq 3 \times 10^{10} \text{ Hz} because there is a suppression in the high frequency range due to the $\coth^{1/2} \left( \frac{k}{2T} \right)$ term panel (a), (figure 1). For comparison, the amplitudes of the spectra are plotted for the decelerated and accelerated universe, (see panel (b), figure 1) with frequency range $\nu_s \leq \nu \leq \nu_E$, where $\nu_s = 10^{-19} \text{ Hz}$. The study shows that the amount of enhancement of the amplitude of the spectrum exceeds the upper bound of WMAP-7 due to $10.01 \text{ Mpc}^{-1}$ while this does not happen for the upper bound of WMAP-5. This amount is independent of the model of expansion of the universe viz accelerated or decelerated, and hence it is due to the thermal nature of the gravitational waves. Further, it is observed that position of the peak at lower frequency remains the same at $\nu_E$ for the accelerated universe (as pointed out in [11] for the vacuum case) but gets enhanced in the thermal vacuum state of gravitational waves. This enhancement is the new feature of the spectrum of the relic gravitational waves in the lower frequency range $\nu_s \leq \nu \leq \nu_E$ (see panel (b), figure 1). And it is the interesting range for the relic gravitational waves on the CMB anisotropy.

The spectral energy density parameter $\Omega_g$ of gravitational waves is defined through the relation

$$\frac{\rho_g}{\rho_c} = \int \Omega_g(\nu) \frac{d\nu}{\nu},$$

where $\rho_g$ is the energy density of the gravitational waves and $\rho_c$ is the critical energy density. Thus

$$\Omega_g(\nu) = \frac{\pi^2}{3} \beta^2 (k, \eta_0) \left( \frac{\nu}{\nu_0} \right)^2,$$

(37)

There is one more consistency condition to be satisfied. Since the space-time is assumed to be spatially flat with $\Omega = 1$, the fraction density should be less than 1, $\frac{\rho_g}{\rho_c} < 1$. Using the normalization of $A$ in eq. (33), we integrate $\int \Omega_g(\nu) \frac{d\nu}{\nu}$, from the $10^{-19} \text{ Hz}$ up to the frequency $\nu_1 = 3 \times 10^{10} \text{ Hz}$, with $\beta = -1.9$ and $\beta_s = -0.552$. The integral is evaluated for $T = 0$, $T = 0.001 \text{ Mpc}^{-1}$ in [23]. For comparison, it is reevaluated with the higher temperature $T = 0.01 \text{ Mpc}^{-1}$ and results are showed for three temperatures $0$, $0.001$, $0.01 \text{ Mpc}^{-1}$ as seen in the following details:

a) $\nu_s \leq \nu \leq \nu_E$,

$$\frac{\rho_g}{\rho_c} = 5.8 \times 10^{-11}, \quad T = 0,$$

$$\frac{\rho_g}{\rho_c} = 8.8 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1},$$

$$\frac{\rho_g}{\rho_c} = 2.6 \times 10^{-10}, \quad T = 0.01 \text{ Mpc}^{-1},$$

b) $\nu_E \leq \nu \leq \nu_H$,

$$\frac{\rho_g}{\rho_c} = 2.3 \times 10^{-11}, \quad T = 0,$$

$$\frac{\rho_g}{\rho_c} = 3.5 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1},$$

$$\frac{\rho_g}{\rho_c} = 1.1 \times 10^{-10}, \quad T = 0.01 \text{ Mpc}^{-1},$$
Figure 2. (color online) The spectral energy density of the gravitational waves for the accelerated (solid lines) and decelerated (dashed lines) universe, panel (a) entire spectrum and panel (b) for \(10^{-19} \text{ Hz} \leq \nu \leq 1.49 \times 10^{-16} \text{ Hz}\).

c) \(\nu_H \leq \nu \leq \nu_2\),
\[
\frac{\rho_g}{\rho_c} = 2.4 \times 10^{-11}, \quad T = 0,
\]
\[
\frac{\rho_g}{\rho_c} = 3.7 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1},
\]
\[
\frac{\rho_g}{\rho_c} = 1.2 \times 10^{-10}, \quad T = 0.01 \text{ Mpc}^{-1},
\]
d) \(\nu_2 \leq \nu \leq \nu_s\),
\[
\frac{\rho_g}{\rho_c} = 8.97 \times 10^{-11}, \quad T = 0,
\]
e) \(\nu_s \leq \nu \leq \nu_{1}\),
\[
\frac{\rho_g}{\rho_c} = 2.7 \times 10^{-6}, \quad T = 0,
\]
It is to be noted that in (d) and (e) stages, the thermal cases are not shown because the thermal contribution in the high frequency range is negligible, due to the temperature dependent term. The combined results are plotted in figure 2. Further, see that the contribution to \(\frac{\rho_g}{\rho_c}\) from the low frequency range is \((10^{-11} - 10^{-10})\) while from the higher frequency range it is \((10^{-6})\). Since the order of contribution to the total \(\frac{\rho_g}{\rho_c}\) from the lower frequency side is very small in contrast with higher frequency side, we get
\[
\frac{\rho_g}{\rho_c} = 2.7 \times 10^{-6}, \quad \nu_s \leq \nu \leq \nu_{1},
\]
which is the same as that of the non-zero temperature case and does not exceed the upper bound put by the nucleosynthesis rate. However, \(\frac{\rho_g}{\rho_c}\) of the gravitational waves with \(T \neq 0\) is higher than the zero temperature case at lower frequency range \(\nu_{1} \leq \nu \leq \nu_{2}\). Therefore, we expect an enhancement for the spectral energy density in the thermal vacuum state in the frequency range \(\nu_{s} \leq \nu \leq \nu_{2}\) only (see panel (b), figure 2). And actually this is the range of interest on the observation point of view of the relic gravitational waves.

Hence by using eq. (37), the allowed amounts of \(\frac{\rho_g}{\rho_c}\) in (a), (b) and (c) stages depend on the upper bound of WMAP-5 and WMAP-7, in figure 1.

Therefore to observe the thermal effect on gravitational waves in this range, we must analyse them separately; otherwise, the signature of the thermal effect gets suppressed due to the contributions from higher frequency range.

3. Conclusion
The gravitational waves are generated during the very early evolution stages of the universe as well as from the various astrophysical objects. Therefore, frequency ranges of the waves vary very widely. There are many ongoing experiments to detect these waves and the range of frequency of interest is from \(10^{-19} \text{ Hz}\) to \(10^{10} \text{ Hz}\). The nature of spectrum of the waves to be observed today is dependent on the evolution history of the universe. Before the results of the super novae type I, the current evolution of the universe is used to consider as matter dominated with decelerated expansion. But, according to the \(\Lambda\)CDM concordance model the present universe is supposed to be driven by dark energy resulting from an accelerated phase. If this is the case, then the spectral property of the waves can be studied by taking into account the current acceleration of the
In the present work, as demanded by the model. The spectral energy density of the gravitational waves in the thermal vacuum state and obtained the spectrum for the accelerated as well as decelerated models for two temperatures $T = 0.001 \text{Mpc}^{-1}$ and $T = 0.01 \text{Mpc}^{-1}$. It is found that the peak position of the spectrum at lower side remains the same as required for the accelerated model but gets enhanced due to the thermal effects of the waves. It is also found that the amount of this enhancement exceeds the upper bound of WMAP-7 due to $T = 0.01 \text{Mpc}^{-1}$ while this does not happen for the upper bound of WMAP-5. Further, this enhancement is the new feature of the spectrum if the relic waves exist during the inflation in thermal vacuum state. Similar enhancement is observed for the spectrum of the relic thermal gravitational waves in the decelerated model, but peaked at $v_0$ as demanded by the model. The spectral enhancement occurs only at the lower frequency range due to the present of temperature dependent term. For the same reason, the temperature contribution to the spectrum towards higher frequency range is negligible.

The spectral energy density of the gravitational waves is estimated in thermal vacuum state for the accelerated and decelerated universe for three temperatures $0.001$, $0.01 \text{Mpc}^{-1}$. It is observed that the spectral energy density gets enhanced in the lower range of frequency $(10^{-11} - 10^{-10})$, but from the higher frequency range it is $(10^{-6})$. Although the temperature dependent term has an insignificant effect on the spectral energy density in the higher frequency range, it is several order magnitude high compared to the lower range. Hence, the spectral energy density of the gravitational waves in thermal vacuum state does not exceed the upper bound put by the nucleosynthesis rate. However, the contribution to the total spectral energy density from the lower range for the relic thermal gravitational waves has to be studied independently because the thermal enhancement is maximum in that range. From observational point of view, this is the range in which currently the detection of the relic gravitational waves is planned. We estimated that the allowed amounts of $\frac{\rho_g}{\rho_c}$ correspond to the upper bound of WMAP-5 and WMAP-7. Therefore the amount of enhancement depends on the upper bound of WMAP.

The present study shows that the amount of enhancement of the amplitude of the spectrum and spectral energy density are independent of the model of expansion of the universe viz accelerated or decelerated and hence the thermal effect of the gravitational waves can account for it. Further, the study indicates that the existence of the gravitational waves in thermal vacuum state is not ruled out. It may be possible to test this scenario with the upcoming data from the Planck or other similar missions.

References
11. Y Zhang et al., Class. Quantum Grav. 22 (2005) 13831394.