

Spectrum of mesons and hyperfine dependence potentials

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(Received 13 July 2005 ; in final form 17 March 2006)

Abstract

In most models, mesons consist of quark -antiquark pairs moving in a confining potential. However, it would be interesting to consider the effect of an extra residual interaction by introducing the quark particles which contain a dependent spin and isospin. In the Chiral constituent quark model, the hyperfine part of the potential is provided by the interaction of the Goldstone bosons, which give rise to a spin- and isospin-dependent part that is crucial for the description of the spectrum for energies lower than 1.7 Gev. In this model we have, not only included the confinement potential at large separations but also the color charge as well as hyperfine interaction potentials. This combination of potentials yields meson spectra which are very close to the ones obtained in experiments.

Keywords: meson, spectrum, quark-confinement potential, color charge

1. Introduction

The justification of the success of the naive quark model (NQM) in describing hadron masses is certainly controversial. However it is worth while to investigate to what extent the spectrum of meson works. Doing so, it enables one to understand how the results can be interpreted in terms of a more convincing model. Many theoretically attempts have been made to compute meson mass with a confining potential. The meson spectrum is usually described by various constituent quark models (CQM). The main point in our model is that not only the confining potential is characterized by presence of long range part confinement, but also by the short-range potential, which is a Coulombic one, that depends on the color charge. Extra hyperfine dependent interquark potential, which contains spin-dependent $H_s(x)$, isospin dependent $H_I(x)$ and spin isospin dependent $H_{sI}(x)$, are also important [1, 2].

The complete interaction used, in this model is given by [3]:

$$H(x) = V(x) + H_s(x) + H_I(x) + H_{sI}(x), \quad (1)$$

where $V(x)$ is the confining potential, and this problem has been solved in [4]. In our model, the confining potential, is a more realistic one, with the added spin and isospin dependent potential $H_s + H_I + H_{sI}$, these residual interaction potential is more important, in the quark interaction [1, 2]. A similar behavior holds for baryons spectrum by using an accurate hypercentral

approximation for solving the nonrelativistic three-body equation describing baryons [5, 6].

2. Interaction potentials

The confining potentials could be of any form (e.g, log, power law, etc). In many practical applications a harmonic oscillator (H.O.) potential yields spectra not much different from those found from potentials such as Coulombic plus linear that QCD prejudice would flavor [7, 8]. Since harmonic oscillator models have nice mathematical properties, they have often been employed as the confining potential, Isgur and Karl [7]. On the other hand, the Coulombic term alone is not sufficient because it would allow free quarks to escape from the system. In this article, the potential is taken as a combination of the Coulombic-like term plus a linear confining term $(ax - c/x)$, as employed by QCD [8,9]. Here we have added the H.O potential which has a two-body character. We have built up a potential scheme for the internal meson structure which has two-body forces between quark and antiquark. Our model is a combination of lattice QCD calculations plus Isgur-Karl [7] interaction. The Isgur-Karl model is an important example of the potential approach to the baryon internal structure. One of its features is the diagonalization in an analytical (H.O.) basis. However, there are now many calculations solving the three-quark Schrödinger equation numerically. To this end the q-q potential is usually derived from the (heavy) quark-antiquark potential [10], since the colour dependence gives

$V_{qq} = \frac{1}{2}V_{q\bar{q}}$. A widely used expression can be found in references [11, 12, 13]

$$V = \sum_{ij} \left(\frac{1}{2} Kr_{ij}^2 + \frac{\alpha_s}{r_{ij}} + br_{ij} \right) + c,$$

where $x_{ij} = r_{ij} = \vec{r}_i - \vec{r}_j$ is the distance between $q\bar{q}$ pair.

Where both the Coulomb-like and linear confinement terms are present, in agreement with the analysis of the meson spectrum. The subtraction of the H.O part simply states that the wave function is used as a convenient mathematical basis. Where the fit was performed using the constants in the potential as a free parameter, the results were not much different from the Isgur-Karl model. In this work we have added the confining hyperfine interaction potentials ($H_s(x)$, $H_I(x)$ and $H_{SI}(x)$). Which yield spectrum very close to the experimental results. By regarding $V(x)$ as the nonperturbative potential and the other terms in eq. (1) as perturbative ones according to this explanation. Where the nonperturbative confining interaction potential now becomes

$$V(x) = ax^2 + bx - c/x, \quad (2)$$

the strength potential parameters a , b and c are constants. This potential has interesting properties since it can be solved analytically, with a good correspondence with physical results. Here the purpose is to use the Schrödinger equation to produce quark wave function in the nonperturbative potential (2).

The spin and isospin potential contains a δ -like term, an ad-hoc operator term [3]. We have modified it by a Gaussian function of the quark pair relative distance

$$H_S = A_S \exp(-x^2 / \sigma_I^2) (s_1 \cdot s_2), \quad (3)$$

where s_i is the spin operator of the i -th quark and $x = r_2 - r_1$ is the relative quark pair coordinate. Furthermore, we add two hyperfine interaction terms to the Hamiltonian quark-antiquark pairs similar to eq. (3). The first one depends on the isospin only and has a form analogous to [3]:

$$H_I = A_I \exp(-x^2 / \sigma_I^2) (t_1 \cdot t_2), \quad (4)$$

where t_i is the isospin operator of the i -th quark. The second one is a spin-isospin interaction, given by [3]

$$H_{SI} = A_{SI} \exp(-x^2 / \sigma_{SI}^2) (s_1 \cdot s_2) (t_1 \cdot t_2), \quad (5)$$

where s_i and t_i are the spin and isospin operators of the i -th quark respectively. The spin potential (3) is provided by the interaction with the Goldstone bosons, which gives rise to a spin- and isospin dependent part. This is good for the description of the spectrum of mesons for energies lower than 1.7 GeV [1, 2]. Recently, it has also been pointed out that an isospin dependence of the quark potential can be obtained by means of the quark exchange [14]. More generally, one can expect that the quark-antiquark pair production can lead to an

effective quark interaction containing an isospin dependent term. On the other hand, based on the fact that the constituent quark model does not contain the hyperfine potential this mechanism may be the reason why the low Q^2 behavior of the electromagnetic transition form factors is not reproduced in the CQM [14, 15].

Then from eqs. (3, 4, 5) the hyperfine interaction (perturbation potential) is given by

$$H_{\text{int}}(x) = H_S(x) + H_I(x) + H_{SI}(x). \quad (6)$$

The strength of the hyperfine interaction in eq. (6) has already been determined [3],

$$A_S = 122.75 \text{ fm}, \quad \sigma_S = 0.8 \text{ fm}, \quad A_I = 51.7 \text{ fm} \quad (7)$$

$$\sigma_I = 3.45 \text{ fm}, \quad A_{SI} = -106.2 \text{ fm}, \quad \sigma_{SI} = 2.31 \text{ fm}.$$

In this paper we solve the Schrödinger equation with the confinement potential (2) exactly. Then by using hyperfine interaction as a perturbation potential we try to find the meson masses.

3. Exact solution of the radial Schrödinger equation with confining potentials. (2)

An exact analytical solution of the radial Schrödinger wave equation for unperturbed confined potential $V(x)$, is presented here.

The method that we are going to introduce, can easily be applied to a two-body problem, where each quark (q or \bar{q}) moves in confining potential, related to other with the potential as in eq. (2). Hence the Schrödinger equation becomes

$$-\frac{\hbar^2}{2\mu} \frac{1}{x^2} \frac{\partial}{\partial x} \left(x^2 \frac{\partial \psi_{v,l}(x)}{\partial x} \right) + (V(x) - E_{v,l} + \frac{l(l+1)\hbar^2}{2\mu x^2}) \psi_{v,l}(x) = 0, \quad (8)$$

where $\mu = \frac{m_q m_{\bar{q}}}{(m_q + m_{\bar{q}})}$ is the reduced mass of q and \bar{q} ,

v is the number of nodes, and l is the angular momentum quantum number. If we set $\psi(r) = \frac{1}{r} \phi(r)$, then the eq. (8) reduces to:

$$\phi''(x) + \frac{2\mu}{\hbar^2} (E - V(x) - \frac{l(l+1)}{2\mu x^2} \hbar^2) \phi(x) = 0. \quad (9)$$

Now for the wave function $\phi(x)$ we make an ansatz by assuming $\hbar = c = 1$ [16, 17, 18, 19].

$$\phi(x) = f(x) \exp[g(x)], \quad (10)$$

where

$$f_v(x) = \prod_{i=1}^v (x - \alpha_i^v) \quad v = 1, 2, \dots \quad (11)$$

$$f(x) = 1 \quad v = 0,$$

$$g(x) = -\frac{1}{2}\alpha x^2 + \beta x + \delta \ln x. \quad (12)$$

Using eq. (10) we have,

$$\varphi''(x) = (g''(x) + g'^2(x) + \frac{f''(x) + 2g'(x)f'(x)}{f(x)})\varphi(x). \quad (13)$$

Comparing eqs. (9) and (13) we can write:

$$V_1(x) + \frac{l(l+1)}{x^2} - \varepsilon = g''(x) + g'(x)^2 + \frac{f''(x) + 2g'(x)f'(x)}{f(x)}. \quad (14)$$

First let us take the $v=0$ state in which $f(x)=1$ and $g(x)$ as in eq. (12) which results in

$$a_1 x^2 + b_1 x - \frac{c_1}{x} + \frac{l(l+1)}{x^2} - \varepsilon = \alpha^2 x^2 - 2\alpha\beta x - \alpha(1+2\delta) + \beta^2 + \frac{2\beta\delta}{x} + \frac{\delta(\delta-1)}{x^2}, \quad (15)$$

where the strengths of the potential (2) and energy becomes:

$$a_1 = 2\mu a, b_1 = 2\mu b, c_1 = 2\mu c, \varepsilon = 2\mu E_{v,l}$$

Using eq.(15) with simple calculations and by equating the powers of x on both sides we find the following corresponding energy and potential parameter relations [17, 18, 19],

$$\alpha = \sqrt{a_1}, \beta = -b_1 / 2\sqrt{a_1}, \delta = l+1, \quad (16)$$

$$\varepsilon = \alpha(1+2\delta) - \beta^2,$$

$$b = \frac{\sqrt{2\mu ac}}{l+1} = \frac{\mu\omega}{l+1} c,$$

where ω is given as,

$$\omega = \sqrt{\frac{2a}{\mu}} = \sqrt{\frac{k}{\mu}}, \quad (17)$$

and $k = 2a$ is similar to the spring constant. Then the energy eigenvalue from eq. (16) for the node $v=0$ state and the angular momentum l is

$$E_{0l} = \sqrt{a/2\mu}(2l+3) - b^2/4a = (l + \frac{3}{2})\omega - \frac{\mu c^2}{2(l+1)^2} \quad (18)$$

The constraining conditions cf. eqs (16), (17) and (18), are the restrictions on the coefficients of the potential strength parameters and energy.

From eq. (10) and these constraints, the unperturbed wave function for the $v=0$ state is

$$\psi_{0l}(x) = N_0 x^{l+2} \exp(-\frac{\mu\omega}{2}x^2 - \frac{b}{\omega}x). \quad (19)$$

Second, for the first node ($v=1$), therefor

$f(x) = (x - \alpha_1^1)$ in eq. (10) and $g(x)$ given as in eq (12), eq. (14) is solved again. By repeating this procedure, parameter α_1^1 is found from the constraint equation

$$\alpha_1^1 = \frac{2(l+1)}{2\beta(l+1) + c_1}. \quad \text{Then the relation between the}$$

potential parameters and the coefficients a, b, c and α_1^1 can be extracted as $a_1 = \alpha^2, b_1 = -2\alpha\beta$ and the restriction equation on the coefficients of the potential parameters for the first unperturbed state $v=1$ is

$$c^2 - \frac{b(2l+3)c}{\sqrt{2\mu a}} + \frac{b^2(l+1)(l+2)}{2\mu a} - \sqrt{\frac{2a}{\mu^3}}(l+1) = 0. \quad (20)$$

Now the energy eigenstate for the first node $v=1$ with angular momentum l is

$$E_{1l} = \sqrt{\frac{a}{2\mu}}(2l+5) - \frac{b^2}{4a} = (l + \frac{5}{2})\omega - \frac{b^2}{2\mu\omega}. \quad (21)$$

The unperturbed eigenfunction of the first excited state is

$$\psi_{1l}(x) = N_1 \left[x - \frac{(l+1)\omega}{\mu c\omega - b(l+1)} \right] \times \exp(-\frac{\mu\omega x^2}{2} - \frac{b}{\omega}x), \quad (22)$$

one can similarly continue for other nodes $v=2,3,\dots$. Note that since the energy difference between any two successive nodes is equal to ω , it is an indication of the validation of this approach.

$$\Delta E = E_{1l} - E_{0l} = \sqrt{\frac{a}{2\mu}}(2) = \sqrt{\frac{2a}{\mu}} = \sqrt{\frac{k}{\mu}} = \omega. \quad (23)$$

Also for $a=1$ from eq.(16) $b=1$; the potential in eq.

(2) turns out to be the Coulomb potential $(\frac{c_1}{x} + \frac{l(l+1)}{x^2})$

with its exact energy spectra given as $\varepsilon_v = \frac{c^2}{4(v+l+1)^2}$ where c is an, independent constant,

and the corresponding eigenfunctions are,

$$\varphi_{vl}(x) = N_v f_v(x)^{l+1} \exp(-\frac{b}{\omega}x) \quad v=1,2,\dots$$

For the zero nod $f_v(x) = 1, v=0$, we have

$$\varphi_{0l}(x) = N_0 x^{l+1} \exp(\beta x) = N_0 x^{l+1} \exp(-\frac{b}{\omega}x) = N_0 x^{l+1} \exp(\frac{-c\mu}{l+1}x). \quad (24)$$

It is clear that $f_v(x) = \prod_{i=1}^v (x - \alpha_i^v)$ are equivalent to

Laguerre polynomials [15]. Also, for $b=0$ and $b=0$;

the potential in eq. (2) becomes harmonic oscillator (H.O) potential only $V(x) = (ax^2 + \frac{l(l+1)}{x^2})$ and its exact

energy spectra is given as

$$E_v = \sqrt{\frac{a}{2\mu}}(2v+3) = \frac{1}{2}(2v+3)\omega \quad v=0,1,2,3,\dots \quad (25)$$

where a is the H.O potential strength and is a constant independent of v where the corresponding eigenfunctions are:

Table 1. The mass of the ρ family mesons with the potential parameters $a = 0.98(fm^{-3})$, $b = 1.15(fm^{-2})$, $c = 2.12$.

ρ family	ν	l	Theory (MeV)	Exp (MeV)
$\rho(770)$	0	0	770	770 ± 3
$1/2(a_1 + a_2)$	0	1	1291	1295 ± 15
$\rho_3(1960)$	0	2	1691	1691 ± 5
$\alpha_4(2040)$	0	3	2031	2037 ± 25
$\rho_5(2350)$	0	4	2371	2350 ± 20
$\rho(1600)$	1	0	1610	1590 ± 20
$\rho(2150)$	2	0	2159	2100-2200
$\rho_3(2250)$	1	2	2243	2225 ± 75

$$\varphi_{\nu l}(x) = f_{\nu}(x)x^{l+1} \exp\left(-\frac{1}{2}\alpha x^2\right) = F_{\nu l}(x) \exp\left(-\frac{1}{2}\alpha x^2\right). \quad (26)$$

It is clear that

$$F_{\nu l}(x) = A_{\nu} x^{l+1} \prod_{i=1}^{\nu} (x - \alpha_i^{\nu}), \quad F_{0l}(x) = A_0 x^{l+1} \quad (27)$$

The polynomial $F_{\nu l}(x)$ above is a spherical Hermite polynomial and the eigenfunction for $\nu=0$ with the angular momentum l becomes

$$\varphi_{0l}(x) = A_0 x^{l+1} \exp\left(-\frac{1}{2}\alpha x^2\right) = \quad (28)$$

$$A_0 x^{l+1} \exp\left(-\frac{1}{2}\mu\omega x^2\right)$$

The Schrödinger equation has been solved and the eigenvalues and eigenfunctions $E_{\nu l}$ for nonperturbative interacting potentials are found analytically. Now, by using the hyperfine interaction as a perturbative potential we can try to find the meson masses as follows.

4. Determining the meson masses family

The mesons masses are given by quark, antiquark mass and the eigenenergies $E_{\nu l}$ of the radial Schrödinger equation E , is a function of a, b, m_q and $m_{\bar{q}}$ and the first order energy correction from nonconfining potential $\langle H_{in} \rangle$ can be obtained by using the unperturbed wavefunction eqs. (19 and 22).

The perturbed wavefunction $\psi_{\gamma}(x)$ for the ground state as well as the other states can be written as

$$\psi_{\gamma} = \psi'_{\gamma} + \sum_{\gamma' \neq \gamma} \frac{\langle \psi'_{\gamma'} | H_{in} | \psi_{\gamma} \rangle \psi'_{\gamma'}}{E_{\gamma} - E_{\gamma'}}. \quad (29)$$

Summarizing the results one can obtain the perturbed energy level E , and the first order corrected energy $\langle H_{in} \rangle = \int \psi_{\gamma} H_{in} \psi_{\gamma} d^3x$ due to the perturbation of the system to the first order. By using the above method one can compute the other orders of energy. The meson

Table 2. The masses of the Υ family mesons with $a = 1.568(fm^{-3})$, $b = 1.84(fm^{-2})$, $c = 2.396$.

Υ families	ν	l	Theory (MeV)	Exp (MeV)
$\Upsilon(9460)$	0	0	9460	9460 ± 0.2
$\Upsilon(10023)$	0	1	10023.4	1023.4 ± 0.5
$\Upsilon(10355)$	0	2	10355.5	10355.5 ± 0.5
$\Upsilon(10575)$	0	3	10615	10577 ± 4
$\Upsilon(10860)$	0	4	10842	10865 ± 8
$\Upsilon(11020)$	1	0	11070	11019 ± 9
$\Upsilon(9915)$	2	0	9910	9903 ± 12
$\Upsilon(10255)$	1	2	10255	10261 ± 2

spectrum then becomes the sum of the quark and anti-quark masses and energy of the perturbed system, thus becomes.

$$M_{q\bar{q}} = m_q + m_{\bar{q}} + E_{\nu l} + \langle H_{in} \rangle, \quad (30)$$

$$M_{q\bar{q}} = m_q + m_{\bar{q}} + \sqrt{\frac{a(m_q + m_{\bar{q}})}{2m_q m_{\bar{q}}}} (2l+3) - \frac{b2}{4a} + \langle H_{in} \rangle, \quad (31)$$

which depends on the terms of the constituent quark and antiquark masses m_q and $m_{\bar{q}}$ and the potentials parameters a and b respectively. Again here l is the angular momentum and ν is the number of nodes of the radial wave eigenfunction $\psi_{\nu l}(x)$. The quark masses used for the various meson families [3] are $m_u = m_d = 257\text{MeV}$, $m_s = 501.5\text{MeV}$, $m_c = 1784\text{MeV}$, $m_b = 5202.2\text{MeV}$. In order to fix the potential parameters we try to fit the ρ and Υ families which are mainly sensitive to the long and short range parts of the potential respectively. In tables 1, 2 and 3 the theoretical and experimental masses of (ρ, Υ) and (φ, k) mesons families are shown in our model. First let's use the well known masses of $\rho(770)$ and $\rho_3(1690)$ as input to determine the two coefficients potential parameters a and b . The theoretical and experimental ρ family meson masses are then shown in table 1.

By consideration of table 1 the masses of the ρ family have been investigated for l ranging from 0 to 4 and $\nu = 0, 1, 2, \dots$

To find the spectrum of Υ family the well known masses of $\Upsilon(9460)$ and $\Upsilon(10355)$ have been used as input to determine the potentials strength a , b and c for Υ family mesons masses.

By careful consideration table 2 is in a good agreement with Υ family with l ranging from 0 to 4 and $\nu = 1, 2, \dots$ is observed. To find the spectrum of φ

Table 3. The potential parameters a , b and c for φ and k mesons families are equal to. $a = 1.107(fm^{-3})$, $b = 1.299(fm^{-2})$, $c = 1.631$.

φ and k	ν	l	Theory (MeV)	Exp (MeV)
$\varphi(1019.5)$	0	0	1020	1019.5
$\frac{1}{2}(f_1(1420) - f_2)$	0	1	1468	1474 ± 8
$\varphi_j(1850)$	0	2	1843	1853 ± 10
$\varphi(1680)$	1	0	1703	1685^{+75}_{-15}
$k(898)$	0	0	892	892
$\frac{1}{2}(k_1(1400) + k_2)$	0	1	1415	1416 ± 6
$k_3(1780)$	0	2	1785	1782 ± 4
$k_4(2060)$	0	3	2058	2060 ± 30

and k family the well known masses of $\varphi(1019.5)$ and $k_4(2060)$ are used as input to determine potential strength a, b and c for φ and k family mesons.

In this table a good agreement has been obtained by our model for φ and k quark families.

5. Conclusion

An exact analytical solution for the potential in the form of the confinement potential is presented. The complete interaction including the spin and isospin terms which reproduces the position of the quarks is also considered

References

1. L Ya, Goltzman and D O Riska, *Phys. Rep. C* **268** (1996) 263.
2. L Ya, Goltzman, Z Papp, W plessas, K varga, R F wagenbrunn, *Phys. Rev. C* **57** (1998) 3406.
3. M M Giannini, E Santopinto and A Vassalo, *Progress in Particle and Nuclear Physics* **50** (2003) 263-270.
4. M Fabre, *Fabre de la Ripelle Nuclear Physics A* **497** (1989) 595c-602c.
5. M M Giannini, E Santopinto and A Vassalo, *Nuclear physics A* **699** (2002) 308-311.
6. M M Giannini, E Santopinto and A Vassalo, *Eur. Phys. J. A* **12** (2001) 447-482.
7. N Isgur and G Karel, *Phys. Rev. D* **18** (1978); *Phys. Rev. D* **19** (1978) 2653.
8. S Gunnar, Bali et al., *Phys. Rev D* **62** (2000) 054503.
9. S Gunnar, Bali, *Phys. Rep* **343** (2001) 1.
10. E Eichten et al, *Phys. Rev. D* **17** (1979) 3090-117, *Phys. Rev. D* **21** (1980) 203-32.
11. M Bohm *Z. Phys. C* **3** (1980) 321-8.
12. I M Barbour and D K ponting, *Z. Phys. C* **4** (1980)

eqs (3,4 and 5). As it is shown in tables 1, 2 and 3 the models in [4, 16] have certainly been improved in this research. It have also been demonstrated that it is possible to find a meson mass by a suitable confining and spin, isospin interacting potentials. In the first column of tables 1, 2 and 3 the various mesons are identified by their symbols and nominal masses, in the second column. The associated ν and the quantum numbers l are shown. The experimental data are given in the last column. In all cases, the theoretical and experimental masses are in complete agreement. Comparing the results of tables 1, 2 and 3 with the results reported in [4, 20], one can see that this model gives better results as it not only includes the confinement but also the color charge and hyperfine potential H_{int} . Furthermore, It is interesting to note that our confining and flavor dependent potential model also covers both upsilonium ($b\bar{b}$) and charmonium ($c\bar{c}$) (Kwong, Rosner) [4]. Therefore this model might be used to provide a better understanding of the hypothesized top quark and study of bottomonium as [20,21].

Acknowledgement

The author wishes to thank Professor H. Arfaee of physics Faculty, Sharif Technical University of Tehran and Dr. H. Movahedian. Shahrood University of technology for their interests in this work and for several useful suggestion.

- 119-29.
13. A M Badalayan, *Phys. Lett.* (1988) 267- 70.
14. F Iachello, *Proceedings of the 1st international Conference on Perspectives in hadronic Physics, ICTP Trieste 12-16 May* (1997), edited by S Boffi, C Ciofi degle Atti, M M Giannini (World Scientific, Singapore, 1998) 363.
15. P Geiger, N Isgur, *Phys. Rev. D* **41** (1990) 1595; **44** (1991) 799; *Phys. Rev. Lett.* **67** (1991)1066; *Phys. Rev. D.* **47** (1993) 5050 ; **55** (1997) 299.
16. A A Rajabi, *Indian Journal of pure and applied physics* **41** (2003) 89-94.
17. R S Kaushal, *Annphys Ny* **206** (1991) 90.
18. A A Rajabi, *Few-body systems, springer ver lage Vien* **37** (2005) 197-213.
19. S Oezelik and Simek, M: *Phys. Lett. A* **152** (1991) 145.
20. M Fabre, de la Riple, *Phys. Lett. B* **205** (1988) 97.
21. J P Tarish, Mc H fiedeldey, M Fabre de la Ripplle and P Vander Merwe, *Few-Body systems* **3** (1988) 99.