Analyzing stability of neutron point kinetics equations with nine photo-neutron groups using Lyapunov exponent method

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Abstract
Lyapunov exponent method is one of the best tools for investigating the range of stability and the transient behavior of the dynamical systems. In beryllium-moderated and heavy water-moderated reactors, photo-neutron plays an important role in dynamic behavior of the reactor. Therefore, stability analysis for changes in the control parameters of the reactor in order to guarantee safety and control nuclear reactor is important. In this work, the range of stability has been investigated using Lyapunov exponent method in response to step, ramp and sinusoidal external reactivities regarding six groups of delayed neutrons plus nine groups of photo-neutrons. The qualitative results are in good agreement with quantitative results of other works.

Keywords: photo-neutron, control parameter, delayed neutron, Lyapunov exponent

1. Introduction

In beryllium-moderated and heavy-water-moderated reactors, photo-neutrons play a direct role in reactor kinetics [1, 2]. Photo-neutrons are produced outside the fuel by both prompt and delayed gamma rays in two ways [1, 2]. The first way is by gamma reactions (γ, n) which usually have high threshold energies (for $-$ Be 1.66MeV, for D_2O, 2.2MeV). The second way is by photo-fission reactions (γ, f), taking place in heavy isotopes [1, 3, 4]. Distributions of these delayed photo-neutrons will change, only slightly, the total of fraction delayed neutrons [5]. Some of these photo-neutrons are due to very long-lived fission fragment decays, compared with delayed neutrons. Therefore, the photo-neutron periods are generally much longer than the delayed neutron periods [1, 2]. Regarding decay constants of photo-neutron precursors and delayed neutron precursors, they can be classified into 15 groups [6]. In such systems, density of neutrons, delayed neutron precursor and photo-neutron precursor concentration are the most important parameters which are to be studied in connection with safety and the transient behavior of the reactor power [7]. These parameters are affected by reactivity. Therefore, reactor stability directly is affected by reactivity [8, 9]. There are several methods for stability analysis of nuclear reactor having been studied before [3, 10, 11], such as Bode, Nyquist, Routh Hurwitz and Lyapunov second method [3, 12, 13]. Fu [9], Chen [14], Ergen [15] and etc studied stability analysis using Lyapunov second method. Recently, Della et al. [13] has developed a theoretical model to study the stability of Ghana Research Reactor (GHARR-1) for a single group of delayed neutrons taking into consideration thermal hydraulics. Another important method for analyzing and diagnosing instability of nuclear reactors is the spectrum of Lyapunov exponents method, that is based on eigenvalues and eigenvectors of the Jacobian matrix [16–18]. The purpose of the research reported here is to introduce the Mean Lyapunov Exponent (MLE[1]) approach on stability analysis of Neutron Point Kinetic (NPK[2]) equations in nuclear reactors with six delayed neutron groups and nine photo-neutron groups in the presence of step, ramp and sinusoidal reactivities.

2. Mathematical formulations
NPK equations taking multi-group delayed neutron and

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1. Mean Lyapunov Exponents

2. Neutron Point Kinetics

photo-neutron into account are presented here for beryllium-moderated and heavy-water-moderated reactors [1, 6, 19, 20]:

$$\frac{dn(t)}{dt} = \left[\frac{\rho(t,n)-\beta}{\Lambda}\right] n(t) + \sum_{i=1}^{I} \lambda_i^d c_i^d(t) + \sum_{j=1}^{J} \lambda_j^p c_j^p(t)$$

$$\frac{dc_i^d(t)}{dt} = \frac{\gamma^d \beta_i^d}{\Lambda} n(t) - \lambda_i^d c_i^d(t) \quad , \quad i=1,2,...,I$$

$$\frac{dc_j^p(t)}{dt} = \frac{\gamma^p \beta_j^p}{\Lambda} n(t) - \lambda_j^p c_j^p(t) \quad , \quad j=1,2,...,J$$

$$(1)$$

$$\beta = \sum_{k=1}^{K} \beta_k = \sum_{i=1}^{I} \gamma^d \beta_i^d + \sum_{j=1}^{J} \gamma^p \beta_j^p \quad , \quad (2)$$

$$\rho(t) = \frac{k_{eff}-1}{k_{eff}} , \quad (3)$$

where, $n(t)$ is the density of neutron, $c_i^d(t)$ and $c_j^p(t)$ are the $i-th$ and $j-th$ concentration of precursor delayed neutrons and photo-neutrons respectively, γ^d and γ^d are the effective coefficients of delayed neutrons and photo-neutrons with estimated theoretical values of 1.23 and 0.246 for Miniature Neutron Source Reactor (MNSR) respectively [20, 21], β_i^d and β_j^p are the relative fractions of delayed neutron and photo-neutron precursors respectively, ρ is the net reactivity which is the sum of external reactivity (ρ_{ext}) and feedback reactivity (ρ_{ext}) [22], β is the total effective fraction of delayed neutrons and photo-neutrons and l is the prompt neutron generation time. In the presence of temperature feedback effects, reactivity is a function of the neutron density and time; therefore, eqs. (1), (2) and (3) are, a system of stiff coupled nonlinear ordinary differential equations [1]. Here the reactivity feedback from arising temperature is being ignored. The effective external reactivities that have been studied are: step ($\rho = \rho_0$), ramp ($\rho = r t$) and sinusoidal reactivity ($\rho = a \sin(\pi t \tau^{-1})$) [1]. Where, r, a and τ are ramp rate reactivity, amplitude and half-period time of the sinusoidal reactivity respectively. In eq. (5), k_{eff} is effective reproduction factor, and is defined as follows [3, 23]:

$$k_{eff} = \frac{Number\ of\ neutrons\ in\ one\ generation}{Number\ of\ neutrons\ in\ preceding\ generation}. \quad (4)$$

If $k_{eff} < 1$, the number of neutrons decreases from generation to generation. So, the system is also subcritical. The system is supercritical if $k_{eff} > 1$, and critical state takes place for $k_{eff} = 1$ [23,24].

3. Analysis Tools
Lyapunov exponents and entropy measures, on the other hand, can be considered "dynamic" measures of attractors complexity which are called "time average" [25]. Lyapunov exponent is useful for distinguishing

various orbits. Three Lyapunov exponents quantify sensitivity of the system to initial conditions and give a measure of predictability. Lyapunov exponent is a measure of the rate at which the trajectories are separated one from another ([26], [27]). A negative exponent implies that the orbits approach to a common fixed point. A zero exponent means that the orbits maintain their relative positions; they are on a stable attractor. Finally, a positive exponent implies that the orbits are on a chaotic attractor, so the presence of a positive Lyapunov exponent indicates chaos. Even though an m-dimensional system has m-Lyapunov exponents, in most applications it is sufficient to compute only the Lyapunov exponents.

3.1 Computation of Lyapunov Exponents
Lyapunov exponents are defined as follows:
Consider two nearest neighboring points (usually the nearest) in phase space at time 0 and t, with distances of the points in the i - th direction $\|\delta x_i(0)\|$ and $\|\delta x_i(t)\|$, respectively. Lyapunov exponent is then defined through the average growth rate Λ_i of the initial distance,

$$\frac{\|\delta x_i(t)\|}{\|\delta x_i(0)\|} = \exp(\Lambda_i t) \quad , \quad (t \to \infty) \quad (5)$$

or

$$\Lambda_i = \lim_{t \to \infty} \frac{1}{t} \log_2 \frac{\|\delta x_i(t)\|}{\|\delta x_i(0)\|}. \quad (6)$$

In the chaotic region, this demonstrates neighboring points with infinitesimal differences at the initial state suddenly separated from each other in the i-th direction [25]. On the other hand, even if the initial states are nearby, the final states are very different. Hence this phenomenon is sometimes named sensitive dependence on initial conditions [25]. Commonly, Lyapunov exponents (Λ_i) can be extracted by observed signals in the following different methods [28]:
- Based on the opinion of following the time-evolution of nearby points in the state space.
- Based on the estimation of local Jacobi matrices.

The first method is usually called Wolf algorithm [29] and it provides an estimation of the largest Lyapunov exponent only. The second method is capable of estimating all the Lyapunov exponents. Using one of these methods, the Lyapunov exponent is calculated rather than a given control parameter. So, there is a little increase in the value of control parameter and the Lyapunov exponent is calculated for the new control parameter. By continuing this method the Lyapunov exponent spectrum of the point reactor kinetics is plotted versus the control parameter.

4. Results and Discussion
MLE method is applied to the stability analysis of NPK equations with delayed neutrons and photo-neutrons in the presence of step, ramp and sinusoidal reactivities. In this work Λ_i $(i=1,2,...,16)$ is i-th MLE with respect to time. All results started from the equilibrium conditions with:

Table 1. Data of Be, D_2O moderated and ^{235}U fuelled reactors.

Delayed neutron		Photo-neutron of Be		Photo-neutron of D_2O	
$\beta_i^d \times 10^{-3}$	λ_i^d	$\beta_i^p \times 10^{-6}$	λ_i^p	$\beta_i^p \times 10^{-6}$	λ_i^p
0.246	0.0127	20.7	2.265×10^{-2}	65.1	0.27726
1.363	0.0317	36.6	8.886×10^{-3}	20.4	1.691×10^{-2}
1.203	0.115	36.8	3.610×10^{-3}	7.00	4.813×10^{-3}
2.605	0.311	7:453	7.453×10^{-4}	3.36	1.500×10^{-3}
0.819	1.40	3.60	2.674×10^{-4}	2.07	4.279×10^{-4}
0.167	3.87	32.0	6.191×10^{-5}	2.34	1.167×10^{-4}
–	–	2.60	1.591×10^{-5}	0.323	4.376×10^{-5}
–	–	0.38	2.478×10^{-6}	0.103	3.633×10^{-6}
–	–	0.57	6.098×10^{-7}	0.05	6.267×10^{-7}

Table 2. MLE with respect to time for different values of l in the presence of positive step reactivity ($\rho_0 = +0.5\beta$).

	Photo-neutrons of Be				Photo-neutron of D_2O			
Λ	$10^{-3}s$	$10^{-4}s$	$10^{-5}s$	$10^{-6}s$	$10^{-3}s$	$10^{-4}s$	$10^{-5}s$	$10^{-6}s$
Λ_1	0.1729	0.1705	0.1727	0.1729	0.1530	0.1694	42.424	896.70
Λ_2	-0.0005	-0.0005	-0.0005	-0.0005	-0.0006	-0.0006	0.1000	671.48
Λ_3	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0012	881.31
Λ_4	-0.0019	-0.0019	-0.0019	-0.0019	-0.0019	-0.0019	-0.0018	885.41
Λ_5	-0.0026	-0.0026	-0.0026	-0.0026	-0.0027	-0.0027	-0.002ᵒ	887.81
Λ_6	-0.0036	-0.0036	-0.0036	-0.0036	-0.0038	-0.0037	-0.0033	889.51
Λ_7	-0.0040	-0.0042	-0.0043	-0.0040	-0.0045	-0.0043	-0.0044	890.83
Λ_8	-0.0052	-0.0052	-0.0052	-0.0052	-0.0069	-0.0070	-0.0050	891.91
Λ_9	-0.0089	-0.0089	-0.0089	-0.0089	-0.0134	-0.0134	-0.0083	892.82
Λ_{10}	-0.0121	-0.0121	-0.0121	-0.0121	-0.0165	-0.0165	-0.0141	893.61
Λ_{11}	-0.0207	-0.0207	-0.0207	-0.0207	-0.0442	-0.0444	-0.0270	894.30
Λ_{12}	-0.0451	-0.0451	-0.0451	-0.0451	-0.1500	-0.1510	-0.0876	894.88
Λ_{13}	-0.1597	-0.1595	-0.1596	-0.1597	-0.2809	-0.2808	-0.2041	895.39
Λ_{14}	-1.1246	-1.1180	-1.1241	-1.1246	-1.0960	-1.1529	-0.6402	895.55
Λ_{15}	-3.7083	-3.6928	-3.7071	-3.7083	-3.3687	-3.7128	-2.2102	894.97
Λ_{16}	-4.6650	-33.629	-70.580	-754.186	-4.9069	-37.855	-67.299	829.72

$$n_0 = 1, \quad c_{i0}^d = \frac{n_0 \beta_i^d \gamma^d}{l \lambda_i^d}, \quad c_{j0}^p = \frac{n_0 \beta_j^p \gamma^p}{l \lambda_j^p}. \tag{7}$$

Effective coefficients of delayed neutrons and photo-neutrons have taken values of $\gamma^d = 1$ and $\gamma^p = 1$ respectively. According to eq. (2) and table 1, β is 0.006552 and 0.007407 for Be and D_2O, respectively. The data used in the study are reported in table 1 [1, 4]. In the following, each reactivity will be discussed further in one subsection.
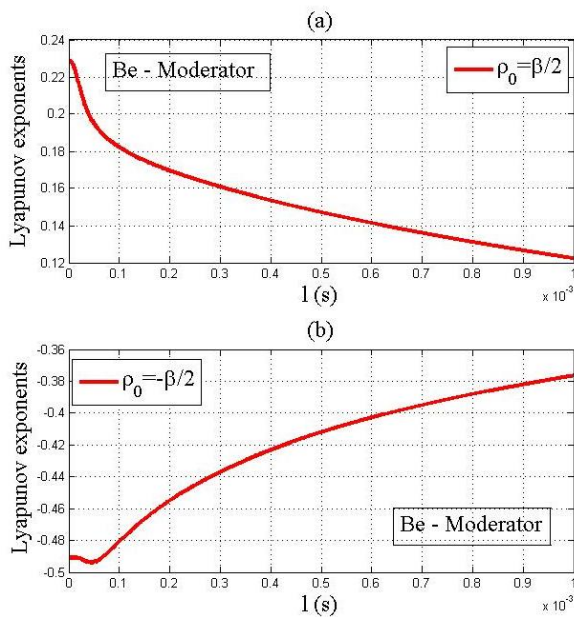
4.1. Step reactivity
Dynamical behavior of NPK equations are studied for different values of the prompt neutron generation time (l). As shown in table 2, when $t \to \infty$, for $\rho_0 = +0.5\beta$, MLEs have positive values. In this situation, the system

is unstable for various values of the prompt neutron generation time; in other words, the nearby trajectories in phase space go away from fixed points. Therefore, predictions are in good agreement with the results of Nahla works [1]. So, density of neutron increases exponentially, and reactor cannot remain in critical state ($k_{eff} = 1.0037$) with this reactivity without taking into account temperature feedback reactivity.
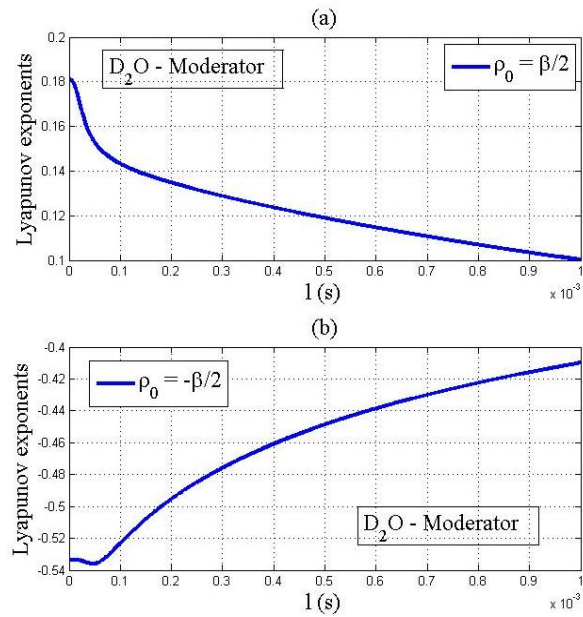
According to table 3, when $t \to \infty$, for $\rho_0 = +0.5\beta$, all MLEs are negative values, therefore, system is asymptotically stable in three dimensional spaces for various values of l, neutron of density decreases exponentially, and reactor goes into subcritical state ($k_{eff} = 0.996$). MLEs with respect to control parameter (l) are shown in figures 1 and 2. As shown in figures 1a and 2a, in a short time interval ($0 \le t \le 20$), MLE will be

Table 3. MLE with respect to time for different values of l in the presence of positive step reactivity ($\rho_0 = -0.5\beta$).

Λ	Photo-neutron of Be				Photo-neutron of D_2O			
	$10^{-3}s$	$10^{-4}s$	$10^{-5}s$	$10^{-6}s$	$10^{-3}s$	$10^{-4}s$	$10^{-5}s$	$10^{-6}s$
Λ_1	-0.0005	-0.0005	-0.0005	-0.0005	-0.0006	-0.0006	-0.0006	-0.0006
Λ_2	-0.0013	-0.0013	-0.0013	-0.0013	-0.0012	-0.0012	-0.0012	-0.0012
Λ_3	-0.0019	-0.0019	-0.0019	-0.0019	-0.0018	-0.0018	-0.0018	-0.0018
Λ_4	-0.0025	-0.0025	-0.0025	-0.0025	-0.0026	-0.0026	-0.0026	-0.0026
Λ_5	-0.0034	-0.0034	-0.0034	-0.0030	-0.0035	-0.0035	-0.0035	-0.0035
Λ_6	-0.0044	-0.0041	-0.0042	-0.0044	-0.0044	-0.0047	-0.0041	-0.0041
Λ_7	-0.0059	-0.0068	-0.0068	-0.0068	-0.0068	-0.0064	-0.0070	-0.0067
Λ_8	-0.0099	-0.0030	-0.0030	-0.0100	-0.0129	-0.0131	-0.0130	-0.0129
Λ_9	-0.0111	-0.0111	-0.0111	-0.0111	-0.0160	-0.0160	-0.0160	-0.0160
Λ_{10}	-0.0199	-0.0202	-0.0110	-0.0202	-0.0230	-0.0231	-0.0230	-0.0232
Λ_{11}	-0.0217	-0.0218	-0.0217	-0.0218	-0.0900	-0.0903	-0.0903	-0.0904
Λ_{12}	-0.0885	-0.0888	-0.0888	-0.0889	-0.2128	-0.2138	-0.2142	-0.2142
Λ_{13}	-0.2268	-0.2279	-0.2281	-0.2281	-0.2826	-0.2826	-0.2826	-0.2827
Λ_{14}	-1.2735	-1.2869	-1.2885	-1.2882	-1.2902	-1.3005	-1.3015	-1.3016
Λ_{15}	-3.7727	-3.8042	-3.8062	-3.8063	-3.7882	-3.8116	-3.8133	-3.8133
Λ_{16}	-10.161	-98.519	-157.48	-170.29	-11.416	-111.31	-152.92	-138.21

Figure 1. Variation of MLE with respect to l ($10^{-6} \leq l(s) \leq 10^{-3}$) for reactors with Be -moderator in the presence of external step reactivity.

Figure 2. Variation of MLE with respect to l ($10^{-6} \leq l(s) \leq 10^{-3}$) for reactors with D_2O -moderator in the presence of external step reactivity.

decreased with increasing l ($10^{-6} \leq l(s) \leq 10^{-3}$) for $\rho_0 = +0.5\beta$. Therefore, exponential growth of the neutron density will be decreased. This result can be seen in table 2 [1]. According to figure 1b and 2b for $0 \leq t \leq 20$, $10^{-6} \leq l(s) \leq 10^{-3}$ and $\rho_0 = -0.5\beta$, all MLEs are in the range of negative. So, the system of behavior is stable in response to a negative step reactivity in duration of 20 seconds, and density of neutron has decreased exponentially. Here, with

increasing l , MLE increases, and exponential decay of the neutron density decreases. These conclusions are corroborated by the results in table 3 [1].
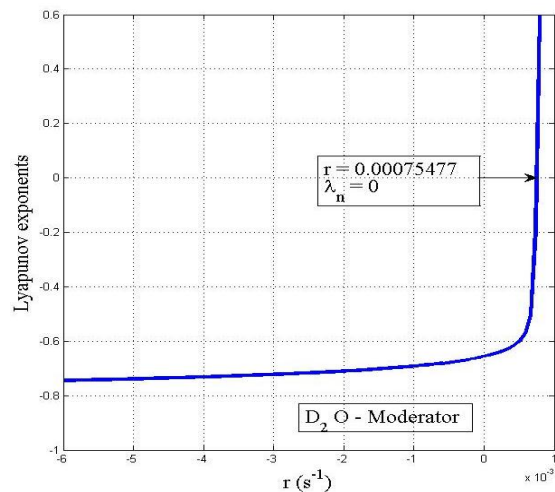
4.2. Ramp reactivity

MLEs with respect to time for ramp reactivity are shown in table 4 when time goes into infinity ($t \rightarrow \infty$). The results imply that, the reactor for all negative ($r \leq 0$) and positive ($r > 0$) values of the ramp rate reactivity are asymptotically stable and unstable respectively. Thus,

Table 4. <u>MLE with respect to time for different values of ramp rate reactivities.</u>

Λ	Photo-neutron of Be				Photo-neutron of D_2O			
	$r = 0.5$	$r = 0.25$	$r = 0.5$	$r = 0.25$	$r = 0.5$	$r = 0.25$	$r = -0.25$	$r = -0.5$
Λ_1	2270.5	1757.1	-0.0025	-0.0025	2194.2	1825.0	-0.0026	-0.0027
Λ_2	1890.9	1388.5	-0.0067	-0.0067	1814.7	1456.9	-0.0066	-0.0070
Λ_3	2239.5	1726.0	-0.0100	-0.0100	2162.8	1795.1	-0.0088	-0.0091
Λ_4	2246.2	1732.5	-0.0117	-0.0117	2169.6	1801.7	-0.0114	-0.0117
Λ_5	2250.2	1736.3	-0.0146	-0.0146	2173.5	1805.6	-0.0141	-0.0143
Λ_6	2253.0	1739.1	-0.0172	-0.0172	2176.3	1808.3	-0.0166	-0.0165
Λ_7	2255.1	1741.1	-0.0203	-0.0203	2178.5	1810.4	-0.0207	-0.0215
Λ_8	2256.9	1742.9	-0.0168	-0.0168	2180.3	1812.1	-0.0207	-0.0205
Λ_9	2258.4	1744.4	-0.0010	-0.0100	2181.8	1813.6	-0.0169	-0.0169
Λ_{10}	2259.7	1745.7	-0.0130	-0.0130	2183.1	1814.9	-0.0237	-0.0226
Λ_{11}	2260.9	1746.8	-0.0203	-0.0203	2184.2	1816.0	-0.1021	-0.1014
Λ_{12}	2261.9	1747.8	-0.1003	-0.1003	2185.2	1817.0	-0.2619	-0.2658
Λ_{13}	2262.8	1748.7	-0.2951	-0.2951	2186.1	1817.9	-0.2943	-0.2915
Λ_{14}	2263.6	1749.5	-1.3829	-1.3829	2186.9	1818.7	-1.3835	-1.3842
Λ_{15}	2264.3	1750.3	-3.8570	-3.8570	2187.7	1819.4	-3.8574	-3.8578
Λ_{16}	2262.3	1745.2	-159.03	-159.03	2185.7	1814.4	-159.01	-159.04
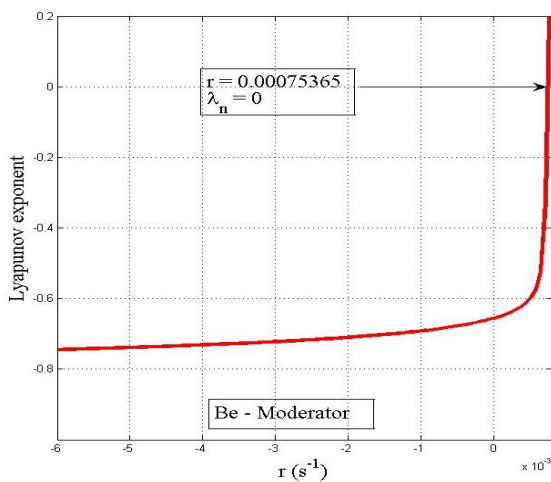
Figure 3. Variation of MLE with respect to the ramp rate reactivities $(-0.006 \leq r(s^{-1}) \leq 0.001)$ for reactors with Be - moderator.

Figure 4. Variation of MLE with respect to the ramp rate reactivities $(-0.006 \leq r(s^{-1}) \leq 0.001)$ for reactors with D_2O - moderator.

For $r \leq 0$, neutron density is reduced gradually over time, and in long term, the reactor is shutdown $(k_{eff} \to 0)$. For $r > 0$, neutron density rises rapidly and reactor period decreases rapidly $(k_{eff} > 1)$. Thus, reactor control will be problematic. Increasing neutron density in table 4 [1] refers to this subject.

Neutron density behavior in response to a ramp reactivity is considered in the duration of 10s. According to figure 3, MLE with respect to ramp rate reactivity will be increased for reactors with Be -moderator with increasing r. For short time intervals, boundary stability in 10s is equal to: $r = 0.00075365$, that is, for $r > 0.00075365$, neutron density will be increased exponentially, so the system is unstable.

In long term, boundary stability tends towards zero value $(r \to 0, k_{eff} \to 1)$. According to figure 4, in reactors with D_2O-moderator, boundary stability in 10s is equal to: $r = 0.00075477$. For each type of reactivity, the range of stability and MLE with respect to control parameters in short time intervals are variable, but in long time, they tend towards a constant value.
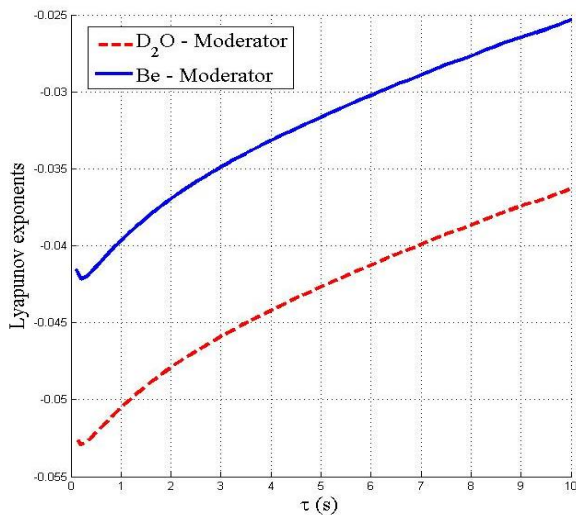
4.3. Sinusoidal reactivity
In this case the reactivity of the system will be applied as follows [1]:

$$\rho(t) = a \sin\left(\frac{\pi t}{\tau}\right), \tag{8}$$

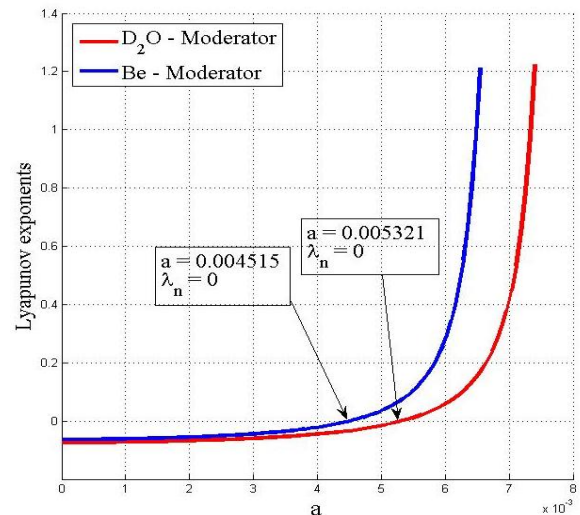**Table 5.** MLE with respect to time for different values of the half-period time of sinusoidal reactivity.

Λ	Photo-neutron of Be				Photo-neutron of D_2O			
	$\tau = 0.1\,s$	$\tau = 1\,s$	$\tau = 10\,s$	$\tau = 100\,s$	$\tau = 0.1\,s$	$\tau = 1\,s$	$\tau = 10\,s$	$\tau = 100\,s$
Λ_1	0.0130	0.0137	0.0186	0.0324	0.0127	0.0133	0.0177	0.0323
Λ_2	-0.0005	-0.0005	-0.0005	-0.0005	-0.0006	-0.0006	-0.0006	-0.0006
Λ_3	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013
Λ_4	-0.0020	-0.0020	-0.0020	-0.0020	-0.0020	-0.0019	-0.0019	-0.0019
Λ_5	-0.0027	-0.0027	-0.002	-0.0027	-0.0028	-0.0028	-0.0028	-0.0028
Λ_6	-0.0038	-0.0037	-0.0037	-0.0037	-0.0039	-0.0039	-0.0039	-0.0039
Λ_7	-0.0042	-0.0043	-0.0041	-0.0042	-0.0043	-0.0043	-0.0043	-0.0045
Λ_8	-0.0052	-0.0052	-0.0052	-0.0053	-0.0079	-0.0072	-0.0073	-0.0073
Λ_9	-0.0090	-0.0090	-0.0090	-0.0090	-0.0139	-0.0139	-0.0139	-0.0141
Λ_{10}	-0.0131	-0.0130	-0.0129	-0.0134	-0.0175	-0.0174	-0.0174	-0.0177
Λ_{11}	-0.0208	-0.0208	-0.0208	-0.0208	-0.0623	-0.0620	-0.0615	-0.0677
Λ_{12}	-0.0623	-0.0621	-0.0614	-0.0673	-0.1746	-0.1745	-0.1755	-0.1827
Λ_{13}	-0.1872	-0.1868	-0.1883	-0.1951	-0.2815	-0.2815	-0.2815	-0.2817
Λ_{14}	-1.2198	-1.2120	-1.2236	-1.2239	-1.2417	-1.2419	-1.2450	-1.2453
Λ_{15}	-3.7667	-3.7675	-3.7683	-3.7683	-3.7779	-3.7786	-3.7793	-3.7793
Λ_{16}	-358.32	-163.67	-157.31	-157.14	-359.41	-164.18	-157.21	-157.33

Figure 5. Variation of MLE with respect to the half-period time of sinusoidal reactivity ($0.1 \le \tau \le 10$) for reactors with Be and D_2O moderators.

Figure 6. Variation of MLE with respect to the amplitude of sinusoidal reactivity ($0 \le a \le 0.0075$) for reactors with Be and D_2O moderators.

where a and $\tau(s)$ are amplitude of reactivity and half-period time of reactivity respectively.

Considering tables 5 and 6 in long term ($t \to \infty$), reactor is unstable for all values of a and $\tau(s)$, that is the neutron density will be increased exponentially ($k_{eff} > 1$). In a short time interval, the reactor can be stable or unstable.

MLEs with respect to control parameters in short time intervals ($0 \le t \le 100$) are shown in figures 5 and 6. Figure 5 shows that in the range of $0.1 \le \tau \le 10$, behavior of the system is stable during 100s from the startup, because in this range of time MLE is negative with respect to control parameter. Figure 6 expresses neutron density behavior in response to changes in the amplitude of sinusoidal reactivity in the range of $0 \le a \le 0.0075$. For reactors with Be - and D_2O - moderators in the range of $a \le 0.0045147$ and $a \le 0.0053212$ respectively, behavior of the system is stable for 100s from the startup. Also, the system is unstable for $a > 0.0045147$ and $a > 0.0053212$ respectively in the same range of time mentioned above.

5. Summary and Conclusion
MLE method is applied to the analysis of stability NPK

Table 6. MLE with respect to time for different values of sinusoidal reactivity amplitude.

Λ	Photo-neutron of Be				Photo-neutron of D_2O			
	$a = \dfrac{\beta}{2}$	$a = \dfrac{\beta}{4}$	$a = \dfrac{\beta}{8}$	$a = 0$	$a = \dfrac{\beta}{2}$	$a = \dfrac{\beta}{4}$	$a = \dfrac{\beta}{8}$	$a = 0$
Λ_1	0.0137	0.0018	0.0002	-0.00003	0.0133	0.0017	0.0002	-0.00007
Λ_2	-0.0005	-0.0005	-0.0006	-0.0007	-0.0006	-0.0006	-0.0006	-0.0007
Λ_3	-0.0013	-0.0013	-0.0014	-0.0014	-0.0013	-0.0013	-0.0013	-0.0014
Λ_4	-0.0020	-0.0020	-0.0021	-0.0021	-0.0019	-0.0019	-0.0020	-0.0020
Λ_5	-0.0027	-0.0027	-0.0029	-0.0030	-0.0028	-0.0028	-0.0028	-0.0028
Λ_6	-0.0037	-0.0038	-0.0038	-0.0038	-0.0039	-0.0040	-0.0041	-0.0041
Λ_7	-0.0043	-0.0043	-0.0041	-0.0043	-0.0043	-0.0044	-0.0043	-0.0045
Λ_8	-0.0052	-0.0053	-0.0053	-0.0053	-0.0073	-0.0074	-0.0076	-0.0074
Λ_9	-0.0090	-0.0091	-0.0091	-0.0091	-0.0139	-0.0142	-0.0143	-0.0143
Λ_{10}	-0.0130	-0.0137	-0.0139	-0.0140	-0.0174	-0.0178	-0.0181	-0.0182
Λ_{11}	-0.0208	-0.0208	-0.0209	-0.0209	-0.0620	-0.0686	-0.0701	-0.0706
Λ_{12}	-0.0621	-0.0682	-0.0696	-0.0700	-0.1744	-0.1820	-0.1841	-0.1843
Λ_{13}	-0.1868	-0.1950	-0.1970	-0.1975	-0.2815	-0.2817	-0.2815	-0.2817
Λ_{14}	-1.2200	-1.2359	-1.2393	-1.2403	-1.2419	-1.2558	-1.2587	-1.2597
Λ_{15}	-3.7675	-3.7763	-3.7782	-3.7788	-3.7786	-3.7865	-3.7882	-3.7887
Λ_{16}	-163.67	-156.73	-156.53	-167.97	-164.18	-157.01	-160.12	-137.76

equations with six delayed neutron groups and nine photo-neutron groups. The influence of step, ramp and sinusoidal reactivities on stability and the neutron density are studied in short time scale and in long-term scale ($t \to \infty$) with MLE method with respect to control parameters. Qualitative results of stability confirm the quantitative results presented in tables 2, 3 and 4 [1]. MLE method is better than traditional stability analysis methods such as, Routh, Nyquist and second methods of Lyapunov, because in Routh and Nyquist methods, finding Laplace transform and poles of characteristic equation are problematic by increasing degrees of freedom systems. Also, in Lyapunov second methods, finding proper Lyapunov function is too hard when dimensions of phase space is being decreased.

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