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Research note

Derivation of master equation for a two-level atom driven by squeezed state field

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Abstract
On one hand a two-level atom is utilized as a qubit in quantum information technology and on the other hand the light in quantum optics is usually in the squeezed state. These reasons motivated us to explore the dynamic of a two-level atom which is driven by a field in the squeezed state. To this goal, the atom operators and the Heisenberg-Langevin equation have been employed. The master equation which reveals the dynamic of the atom has been derived using some features of the squeezed state and doing some algebra. Finally, the dynamic of the two-level atom which is driven by squeezed state field has been simulated by the derived master equation. Also, the effect of atom parameters on the dynamic has been investigated.

Keywords: master equation, two-level atom, squeezed state, SLH framework

1. Introduction

In recent years, developing quantum technology makes researchers and scientists pay more attention to the control of quantum systems. The ability of not only observing but also manipulating quantum physical processes would be a demand result of the quantum control [1]. Because of non-classical characteristics of quantum regime, modeling and control of quantum systems are more challenging. A comprehensive model, which describes the dynamic of the system under control and its interactions with the environment, is a preliminary to control theory. The modeling of a quantum system interacting with the environment is discussed in the open quantum system framework [2]. However, the interaction of a quantum system with different fields, which are considered as the environment, has been investigated, including field in vacuum [3, 4], Gaussian [5], single-photon, and superposition of coherent states [3, 6].

Technology has been developed in decades on the basis of binary numeral system which has just two states zero and one. The basic unit of information in the binary systems is called *bit*. In quantum technology, a quantum bit (Qubit) has been seen as an analogy for the bit. A Qubit, like spin of electron, has two base states which can play the role of information carrier [7, 8]. So, a two-level quantum system is a fundamental component to develop quantum technologies such as quantum computers and its interaction with different fields is interested.

The pure state of a quantum system is denoted by the ket $|\psi\rangle$ in the corresponding Hilbert space $\mathbb{H}$. Yet, a generalized state is represented by the density matrix ρ which in the pure case is $\rho = |\psi\rangle\langle\psi|$. Here $\langle\psi|$ is $\langle\psi| = |\psi\rangle^\dagger$. Any physical measurement of the system is expressed by acting a Hermitian operator X on the system state as $\mathrm{tr}[\rho X]$ where tr denotes the trace. The quantum expectation of time evolution of a quantum system is given by a differential equation, which is named the master equation (ME), for the ρ or operator X. The former and the later representations have been called the Schrodinger and the Heisenberg picture, respectively [9]. The behavior of a two-level atom which is driven by the superposition of coherent state, single photon, and vacuum state have been studied by utilizing the master equation [10].

The light in the quantum optics is frequently considered as a field in the squeezed state [11]. Also, the

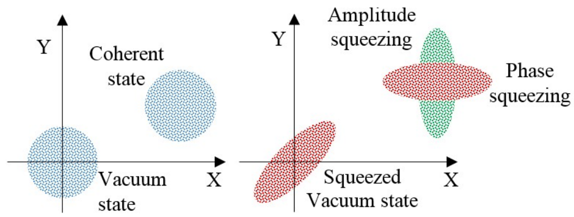

Figure 1. Vacuum state, coherent state, and squeezed states of light.

squeezed state not only plays a significant role in quantum noise theory [12] but also is using in the detection of gravitational waves [13]. The squeezed field has been employed as a control source in [14]. Also, the filtering of a quantum system driven by filed in a squeezed state has been derived in [15]. The inversion of an atom initially in the ground state is given by Cummings in [16]. The squeezed state is considered a superposition of number states which are weighted according to a distribution. A vector representation of the Cumming's solution is represented in [17], where the interaction of a two-level atom with squeezed light has been investigated. Also, the collapse time dependence on the direction of squeezing has been discussed. How the presence of a single two-level atom inside an oscillator affects the degree of squeezing of the light has been studied in [18].

This paper concerns the dynamic of a two-level atom driven by the field in squeezed state. The interaction has been studied in the SLH representation. Thus, first, the Pauli matrices have been selected as the operators of the two-level atom. Second, the Heisenberg-Langevin equation has been employed to derive the ME by quantum expectation over the field. Some simplification has been done by using the characteristics of squeezed state to finally derive the master equation for a two-level quantum system which is driven by the squeezed state field. At the end, the behavior of the atom under foregoing circumstances has been simulated and the effect of parameters has been discussed.

After this brief introduction, section II represents the squeezed field and its features. Time evolution of a quantum system and the Heisenberg-Langevin equation are introduced in section III. The fourth one is devoted to derive the ME for the two-level atom driven by light in the squeezed state. Simulating the derived ME and investigating the effect of different parameters on the dynamic of the system are given in V. At the end, the paper is concluded in VI.

2. Squeezed state field

The area where fundamental physical processes occur at the absolute zero temperature and there is no matter in it, is called the quantum vacuum, Any noise or movement is not expected in the vacuum state but some fluctuations can be seen, which is one of the features of the quantum vacuum state nature [19]. Interestingly, the noise of electric field for the light in squeezed state is lower than the vacuum state at the certain phases. That is to say when the squeezed light is turned on, we see less noise than no light. This paradoxical feature of light cannot be explained by classical rules and is the quantum nature of light [20].

A squeezed field is a photon that one of its canonical observables has less uncertainty than the other. The uncertainty in the measurement of an observable X is the mean variance ΔX. Assuming the state of light is represented by the end point of a complex phasors, then a classical light is a point in the complex plane. But, respecting the Heisenberg's Uncertainty Principle in quantum optics, there is a quantum uncertainty where any measurement of the light field delivers different values within an uncertainty region. Figure.1 shows different squeezed lights.

If $|\psi\rangle$ be an eigenstate of X, i.e., $X|\psi\rangle = x|\psi\rangle$, then the mean variance is zero. But, two non-commuting observables X and Y that $[X,Y] = XY - YX = iZ$ do not have any common eigenstate. As a result, they cannot be determined precisely according to uncertainty principle [21]:

$$\Delta X \Delta Y \geq \frac{1}{2}|\langle Z\rangle| \tag{1}$$

Apparently, ΔX and ΔY are not required to be equal. If $\Delta X < \sqrt{\frac{1}{2}|\langle Z\rangle|}$, the state $|\psi\rangle$ is called squeezed with respect to the observable X. It worth to note that the squeezing is not possible without increasing of fluctuations in the conjugate observable. The squeezed states of light are yielded in non-linear processes in a classical electromagnetic field impelled forward in non-linear environment. The productive Hamiltonian for a nonlinear optical process would be [21]:

$$H = \epsilon(a^\dagger)^2 + \epsilon^* a^2, \tag{2}$$

where ϵ is the amplitude of classical electromagnetic field. a and $a^\dagger$ are the annihilation and creation operators satisfy $[a,a^\dagger] = I; [a, I] = 0; [a^\dagger, I] = 0$. The operator H explains how the frequency of input field is converted to the half of the driving harmonic [21]. Defining the squeezing parameter $\xi = -i\epsilon t/\hbar$, the time evolution of the single-mode radiation $U(t) = \exp(-iHt/\hbar)$, can be written as [20, 21]:

$$S(\xi) = \exp\left\{\xi\frac{\left(a^\dagger\right)^2}{2} - \xi^*\frac{a^2}{2}\right\}, \tag{3}$$

Generally, $\xi = re^{i\phi}$ where ϕ is a real number and $r = \ln(R)$. $R > 0$ is named the squeezing factor. Thus, considering that the electromagnetic mode is initially in the vacuum state $|\xi\rangle(t)=S(t)|0\rangle$, which is commonly known as the squeezed vacuum state. This state could be expressed in the Fock basis as [21]:

$$|\xi\rangle=\frac{1}{\sqrt{\cosh(r)}}\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{2^n n!}\left(e^{i\phi}\tanh(r)\right)^2|2n\rangle, \tag{4}$$

where $|n\rangle$ is the eigenstate of number operator $N=a^\dagger a$.

The Bogoliubov transformation on the operators α and $a^\dagger$ give the annihilation operator b [21]:

$$\begin{aligned} b&=S(\xi)aS^\dagger(\xi)=a\cosh(r)-a^\dagger e^{i\phi}\sinh(r),\\ b^\dagger&=S(\xi)a^\dagger S^\dagger(\xi)a^\dagger\cosh(r)-ae^{-i\phi}\sinh(r)\end{aligned} \tag{5}$$

The eigen-decomposition of the annihilation operator b results in eigenstate and eigenvalues as the squeezed vacuum state $|\xi\rangle$ and zero, respectively. That is to say, the squeezed vacuum state of the operator a is equivalent to the ground state of the transformed oscillator b. It worth to note that defining $n=\langle a^\dagger a\rangle$ and $m=\langle a^2\rangle$ for a quantum Gaussian states, we have $|m|^2\le n(n+1)$ [22]. The vacuum state is specified by the choice $n=0$ and $m=0$.

3. System evolution

In the SLH method, any quantum component has been explained by $G=(S,L,H)$ which displays the interaction of the internal degrees of freedom with the input field. H is the Hamiltonian, L represents how the system is coupled with the external field, and S indicates the scattering of the input/output channels. The time evolution of such system is given by a quantum stochastic differential equation (QSDE), which is known as Hudson-Parthasarathy equation, as [12, 15]:

$$dU(t)=\begin{Bmatrix}-(iH+K)dt+LdA^\dagger\\ -L^\dagger SdA+(S-I)d\Lambda\end{Bmatrix}U(t) \tag{6}$$

With $U(0)=I$, where the operator K is given by

$$K=\frac{1}{2}\begin{pmatrix}dAdA^\dagger L^\dagger L+dA^\dagger dALL^\dagger-dAdA(L^\dagger)^2\\ -dA^\dagger dA^\dagger(L)^2\end{pmatrix}. \tag{7}$$

The scattering process $dA(t)$ is an increment in the field's number operator. $dA(t)$ and $dA^\dagger(t)$, the annihilation and creation processes, have been defined as the increments of time-integrated quantities of the input field

$$dA(t)=\int_t^{t+dt}a(s)ds,\ dA^\dagger(t)=\int_t^{t+dt}a^\dagger(s)ds, \tag{8}$$

which satisfy $\left[dA(t),dA^\dagger(t')\right]=\delta(t-t')dt$ where δ is the Kronecker function. We note that operators $a(t)$ and $a^\dagger(t)$ are the annihilation and creation operators of the input field which satisfy $\left[a(t),a^\dagger(t')\right]=\delta(t-t')$.

Considering the quantum Ito table for squeezed field as

$$\begin{aligned} dAdA^\dagger&=(1+n)dt, & dA^\dagger dA&=ndt,\\ dAdA&=mdt, & dA^\dagger dA^\dagger&=m^\dagger dt,\end{aligned} \tag{9}$$

then, the operator K in Eq. (6) becomes $K_{sq}=\frac{1}{2}\left((n+1)L^\dagger L+nLL^\dagger-m\left(L^\dagger\right)^2-m^\dagger L^2\right)$. Also, considering a squeezed state which has no scattering, that is to say the squeezed field radiates totally to the system and do not scatter to the bath ($S=I$), we have $dU(t)=\{-(iH+K_{sq})dt+LdA^\dagger-L^\dagger dA\}U(t)$ [15].

Considering the joint $system\otimes field$ state, the time evolution for a given system operator X is $j_t(X)\equiv U^\dagger(t)[X\otimes I]U(t)$. Applying the Ito differentiating rule, $d(AB)=(dA)B+A(dB)+(dA)(dB)$, using the Ito table (9), and doing some simplification leads to:

$$\begin{aligned} dj_t(X)&=j_t(-i[X,H]+\mathcal{L}X)dt\\ &+j_t([X,L])dA^\dagger+j_t\left(\left[L^\dagger,X\right]dA\right),\end{aligned} \tag{10}$$

where

$$\begin{aligned}\mathcal{L}X=&\frac{1}{2}\left\{L^\dagger[X,L]+\left[L^\dagger,X\right]L\right\}+\frac{n}{2}\begin{Bmatrix}L^\dagger[X,L]+\left[L^\dagger,X\right]L\\ +L\left[X,L^\dagger\right]+[L,X]L^\dagger\end{Bmatrix}\\ &-\frac{m^*}{2}\{L[X,L]+[L,X]L\}-\frac{m}{2}\left\{L^\dagger\left[X,L^\dagger\right]+\left[L^\dagger,X\right]L^\dagger\right\}.\end{aligned}$$

Now, the dynamic of the observation X could be written down by tracing (Expectation $\mathbf{E}$) over the field on eq. (10). For Gaussian states inputs we have $\mathbf{E}[dA]=\alpha(t)dt$. So, the master equation would be

$$\begin{aligned}\dot{\varpi}_t(X)=&\varpi_t\{i[X,H]\}\\ &+\varpi_t\left\{\frac{1}{2}L^\dagger[X,L]\frac{1}{2}\left[L^\dagger,X\right]L\right\}\\ &+\varpi_t\{[X,L]\}\alpha^*(t)+\varpi_t\left\{\left[L^\dagger,X\right]\right\}\alpha(t)\\ &+\varpi_t\left\{\frac{n}{2}L^\dagger[X,L]\right\}+\varpi_t\left\{\frac{n}{2}\left[L^\dagger,X\right]L\right\}\\ &+\varpi_t\left\{\frac{n}{2}L\left[X,L^\dagger\right]\right\}+\varpi_t\left\{\frac{n}{2}[L,X]L^\dagger\right\}\\ &-\varpi_t\left\{\frac{m^*}{2}L[X,L]\right\}-\varpi_t\left\{\frac{m^*}{2}[L,X]L\right\}\\ &-\varpi_t\left\{\frac{m}{2}L^\dagger\left[X,L^\dagger\right]\right\}-\varpi_t\left\{\frac{m}{2}\left[L^\dagger,X\right]L^\dagger\right\}.\end{aligned} \tag{10}$$

where $\varpi_t(X)$ denotes the average of an observable X of the system at time t.

4. Tow-level atom driven by squeezed state: master equation

The SLH model of a two-level atom is [10]:

$$G=(I,\sqrt{\gamma}\sigma_-,\frac{\omega}{2}\sigma_z), \tag{11}$$

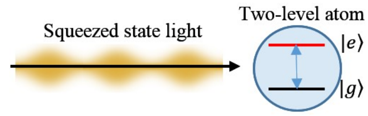

Figure 2. A two-level atom driven by squeezed light.

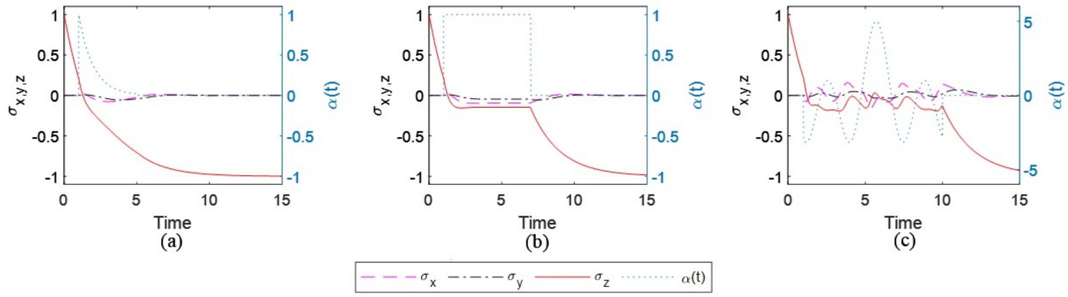

Figure 3. Average time evolution of the atom.

where $\sigma_{\pm}=\frac{1}{2}(\sigma_x \pm i\sigma_y)$ and $\sigma_{x,y,z}$ are the Pauli matrices. The excited $|e\rangle$ and the ground $|g\rangle$ states are two base states of a two-level quantum system. Assume an atom is driven by the field in the squeezed state, see figure.2. The dynamic of the atom in this situation can be investigated by deriving the master equations for the σ_x, σ_y, and σ_z operators as follow.

Assume a two-level atom represented in SLH framework by Eq. (12) which is driven by the field in the squeezed state. Then, the dynamic of the atom is:

$$\begin{aligned}
\dot{\varpi}_t(\sigma_x) &= \sqrt{\gamma}\left(\alpha(t)+\alpha^*(t)\right)\varpi_t(\sigma_z)-0.5\gamma\left(1-2n-m-m^*\right) \\
&\quad \varpi_t(\sigma_x)-0.5\left(\gamma m-\gamma m^*+2\omega\right)\varpi_t(\sigma_y), \\
\dot{\varpi}_t(\sigma_y) &= i\sqrt{\gamma}\left(\alpha(t)-\alpha^*(t)\right)\varpi_t(\sigma_z) \\
&\quad -\gamma\left(\frac{1}{2}-2n\right)\varpi_t(\sigma_y)+\omega\varpi_t(\sigma_x), \\
\dot{\varpi}_t(\sigma_z) &= -\sqrt{\gamma}\left(\alpha(t)+\alpha^*(t)\right)\varpi_t(\sigma_x) \\
&\quad -i\sqrt{\gamma}\left(\alpha(t)-\alpha^*(t)\right)\varpi_t(\sigma_y) \\
&\quad -2\gamma n\varpi_t(\sigma_z)-\sqrt{\gamma}\left(1+\varpi_t(\sigma_z)\right).
\end{aligned} \tag{12}$$

These equations can be derived by substituting the system parameters eq. (12) and the Pauli matrix σ_x, σ_y, and σ_z into (11). For instance, considering σ_x results in

$$\begin{aligned}
\dot{\varpi}_t(\sigma_x) &= -i\frac{\omega}{2}\varpi_t(\sigma_x\sigma_z-\sigma_z\sigma_x) \\
&\quad +\frac{\gamma}{2}\varpi_t(\sigma_+[\sigma_x,\sigma_-])+\frac{\gamma}{2}\varpi_t([\sigma_+,\sigma_x]\sigma_-) \\
&\quad +\sqrt{\gamma}\alpha^*(t)\varpi_t([\sigma_x,\sigma_-])+\sqrt{\gamma}\alpha(t)\varpi_t([\sigma_+,\sigma_x]) \\
&\quad +\frac{\gamma n}{2}\varpi_t(\sigma_+[\sigma_x,\sigma_-])+\frac{\gamma n}{2}\varpi_t([\sigma_+,\sigma_x]\sigma_-) \\
&\quad -\frac{\gamma n}{2}\varpi_t(\sigma_-[\sigma_x,\sigma_+])+\frac{\gamma n}{2}\varpi_t([\sigma_-,\sigma_x]\sigma_+) \\
&\quad -\frac{\gamma m^*}{2}\varpi_t(\sigma_-[\sigma_x,\sigma_-])-\frac{\gamma m^*}{2}\varpi_t([\sigma_-,\sigma_x]\sigma_-) \\
&\quad -\frac{\gamma m}{2}\varpi_t(\sigma_+[\sigma_x,\sigma_+])-\frac{\gamma m}{2}\varpi_t([\sigma_+,\sigma_x]\sigma_+)
\end{aligned} \tag{13}$$

The operators can be represented in the base states as
$\sigma_z=|ee|-|gg|$, $\sigma_y=i(|ge|-|eg|)$,
$\sigma_x=|ge|+|eg|$, $\sigma_-=|ge|$, $\sigma_+=|eg|$.
Substituting these values into eq.(14) and doing some simple algebra using $\langle e|e\rangle=\langle g|g\rangle=1$ and $\langle e|g\rangle=\langle g|e\rangle=0$ results in eq.(13). The equations of σ_y and σ_z can be derived in the same way as σ_x.

5. Simulation results

Let $\omega=1$, $\gamma=0.5$, $n=2$, $m=1$. Assuming that the atom is initially in the excited state, that is to say $\sigma_x=0$, $\sigma_y=0$ and $\sigma_z=1$. In addition, three fictitious field shapes have been considered:

$$\alpha(t)=e^{-(t-1)}\left(u(t-1)-u(t-5)\right) \tag{14}$$

$$\alpha(t)=u(t-1)-u(t-7) \quad (14)$$

$$\alpha(t)=\left(3\cos(2t-2+\pi)+2\cos\left(t-1+\frac{\pi}{2}\right)\right)\left(u(t-1)-u(t-10)\right) \tag{15}$$

The behavior of two-level atom has been depicted in figure.3-(a), (b), and (c) by simulating the ME (13) in time interval 0 to 15 considering field shapes as Eqs. (15), (16), and (17), respectively. It can be seen that the atom dissipates its energy and goes from the excited state to the ground state. At the time interval that the input field radiates to the atom ([1-5], [1-7]), and [1, 10] for (15), (16), and (17), respectively) the reduction rate varies which make sense due to absorbing energy from the field. It can be seen in Figure.3-(a) that the dissipation rate decreases when the field radiates to the atom. Figure.3-(b) shows that the atom converges to a steady value as a steady field emitted to the atom. Furthermore, where the field has a sinusoidal form in Figure.3-(c), the atom state has some fluctuations which is the behavior that has also been seen in [17]. The state has been shown in Bloch sphere in Figure. (4) as well.

In order to reveal the effect of ω on the behavior of the atom, the ME has been simulated for different values

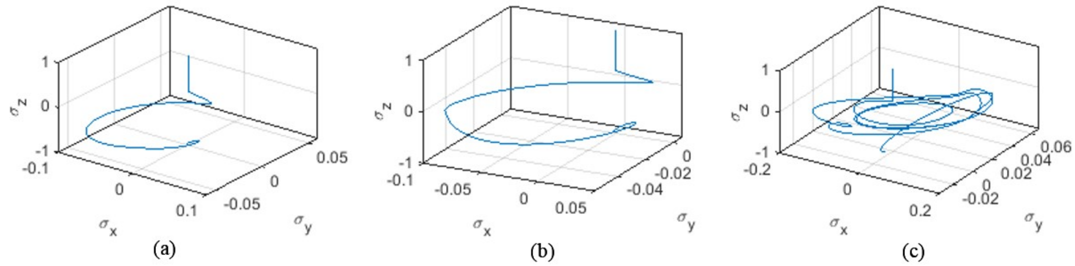

Figure 4. The atom state in Bloch sphere.

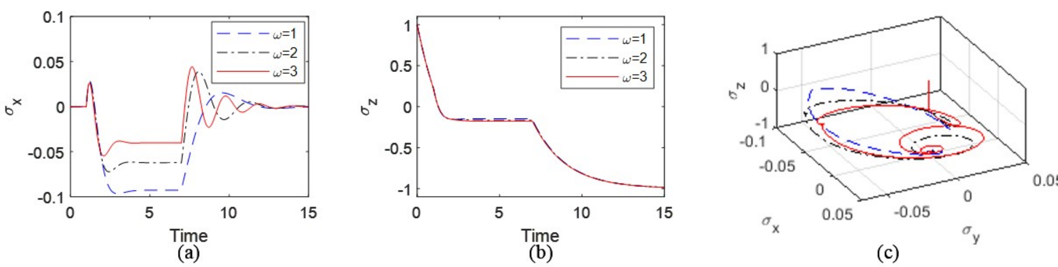

Figure 5. State trajectories for $\gamma = 0.5$ and different values of ω. (a) σ_x; (b) σ_z; (c) in Bloch sphere.

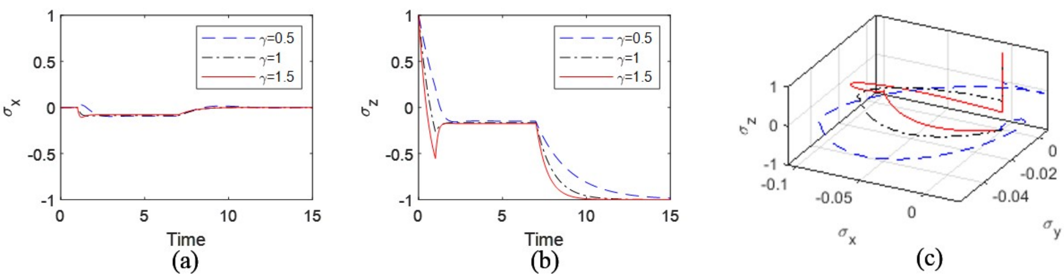

Figure 6. State trajectories for $\omega = 1$ and different values of γ. (a) σ_x; (b) σ_z; (c) in Bloch sphere.

of ω considering $\gamma = 0.5$ and the field shape of Eq. (16). The results have been shown in Figure. 5. The variation of ω has no significant effect on the σ_z (Figure.5-(b)) but increases the oscillation frequency of σ_x and σ_y (Figure.5-(a)). This means that the state rotates faster in a smaller area, which is shown in Figure. 5- (c), yet the rate of energy dissipation does not change.

The effect of γ on the behavior of the atom is explored by plotting the ME for $\omega = 1$ and different values of γ. The results which have been plotted in Figure. 6 show that the higher the coupling γ, the faster the dissipation to the ground state which can be seen in Figure.6-(b). The faster movement of the trajectories which are depicted in Figure.6-(c) confirms this effect. Also, it demonstrates that the atom absorbs energy from the field in time interval t=1 to 7 which the squeezed field radiates to the atom. Figure.6-(a) shows that the change of γ has no meaningful effect on σ_x or σ_y.

6. Conclusion

The behavior of a two-level atom which is driven by field in squeezed state was explored in this paper. The ME of the Pauli matrices (as atom's operators), which represent the dynamic of the atom, was derived. The simulation of the derived ME showed that the atom dissipates its energy into the bath while absorbing energy when the squeezed field radiates to it. In addition, the coupling constant γ determines the rate of dissipation and absorption.

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