# Entanglement Amplification by Three-Level Laser Coupled to Vacuum reservoir

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**Abstract**

In this paper we have studied the squeezing and entanglement properties of the cavity light generated by a three-level laser. In this quantum optical system, N three-level atoms available in an open cavity, coupled to a two-mode vacuum reservoir, are pumped to the top level by means of electron bombardment at constant rate. Applying the solutions of the equations of evolution for the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators, we have calculated the mean, variance of the photon number, the quadrature squeezing, entanglement amplification as well as the normalized second-order correlation function for the cavity light. In addition, we have shown that the presence of the spontaneous emission process leads to a decrease in the mean and variance of the photon number. We have observed that the two-mode cavity light is in a squeezed state and the squeezing occurs in the minus quadrature. In addition, we have found that the effect of the vacuum reservoir noise is to increase the photon-number variance and to decrease the quadrature squeezing of the cavity light. However, the vacuum reservoir noise does not have any effect on the mean photon number. Moreover, the maximum quadrature squeezing of the light generated by the laser, operating far below threshold, is found to be 37.5% below the vacuum-state level. In addition, our result indicates that the quadrature squeezing is greater for $\gamma = 0$ than that for $\gamma = 0.4$ for $0.01 < r_a < 0.35$ and is smaller for $\gamma = 0$ than that for $\gamma = 0.4$ for $0.35 < r_a < 1$. We have also noted that the squeezing and entanglement in the two-mode light are directly related. As a result, an increase in the degree of squeezing directly leads to an increase in the degree of entanglement and vice versa. This shows that, whenever there is squeezing in the two-mode light, there exists an entanglement in the system.

**Keywords:** Master Equation, Photon statistics, Quadrature squeezing, Spontaneous emission, second-order correlations, photon entanglement

## 1. Introduction

Entanglement is one of the fundamental tools for the quantum information processing and communication protocols. The generation and manipulation of the entanglement has attracted a great deal of interest with wide applications in quantum teleportation, quantum dense coding, quantum computation, quantum error correction, and quantum cryptography [1-5]. Recently, much attention is given to the generation of a continuous-variable entanglement to manipulate the discrete counterparts and quantum bits and to perform the quantum information processing. In general, the degree of entanglement decreases, when it interacts with the environment. But, the quantum information processing efficiency highly depends on the degree of entanglement. Therefore, it is necessary to generate strongly entangled states which can survive under the external noise. In general, due to the strong correlation between the cavity modes, a two-mode squeezed state violates certain classical inequalities and then can be used in preparing the Einstein–Podolsky–Rosen (EPR)-type entanglement [6]. The steady state entanglement in a nondegenerate three-level laser has been studied, when the atomic coherence is induced by initially preparing atoms in a coherent superposition of the top and bottom levels [7-15] and when the top and bottom levels of three-level atoms injected into a cavity are coupled by coherent light [16-21].

Recently, Menisha [17] has studied the squeezing and the statistical properties of the light produced by a three-level laser with the atoms placed in an open cavity and pumped by electron bombardment. He has shown that the maximum quadrature squeezing of the light generated by the laser, operating below threshold, is found to be 50%

below the vacuum-state level. In addition, Fesseha [10] has studied the squeezing and the statistical properties of the light produced by a three-level laser with the atoms placed in a closed cavity and pumped by coherent light. He has shown that the maximum quadrature squeezing is 43% below the vacuum-state level, which is slightly less than the result found with electron bombardment. He has also found that a large part of the total mean photon number is confined in a relatively small frequency interval.

In this paper, we seek to analyze the squeezing and entanglement properties of light emitted by three-level atoms available in an open cavity and pumped to the top level by electron bombardment. Thus taking into account the interaction of the three-level atoms with a resonant cavity light and the damping of the cavity light by a vacuum reservoir, we obtain the photon statistics, the quadrature squeezing, entanglement, and the normalized second-order correlation function for the cavity light. We carry out our calculation by considering the interaction of the three-level atoms with the vacuum reservoir outside the cavity light.

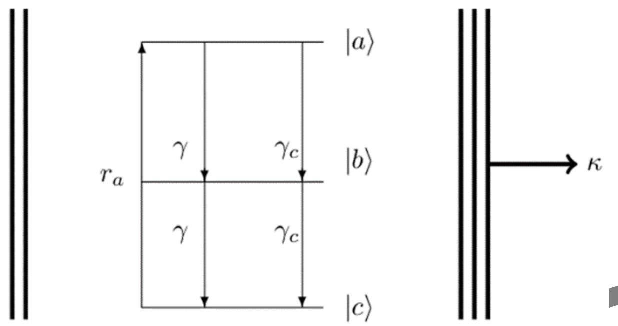



Figure 1: Schematic representation of a three-level laser coupled to a two-mode vacuum reservoir.

## 2. Master Equation

We consider here the case in which N three-level atoms in a cascade configuration and available in an open cavity. We denote the top, intermediate, and bottom levels of these atoms by$|a\rangle_k$, $|b\rangle_k$, and $|c\rangle_k$, respectively. We prefer to call the light emitted from the top level light mode $a$ and the one emitted from the intermediate level light mode$b$. We carry out our analysis with light modes $a$ and $b$ having the same or different frequencies. In addition, we assume that light modes $a$ and $b$ to be at resonance with the two transitions $|a\rangle_k \rightarrow |b\rangle_k$ and$|b\rangle_k \rightarrow |c\rangle_k$, with direct transition between $|a\rangle_k$ and $|c\rangle_k$ to be electric-dipole forbidden. The interaction of a three-level atoms with cavity modes $a$ and $b$ can be described at resonance by the Hamiltonian [9]

$$\hat{H} = ig\left(\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^{\dagger}\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - \hat{b}^{\dagger}\hat{\sigma}_b^k\right), \quad (1)$$

Where

$$\hat{\sigma}_a^k = |b\rangle_{kk}\langle a|, \quad (2)$$

and

$$\hat{\sigma}_b^k = |c\rangle_{kk}\langle b| \quad (3)$$

The quantum Langevin equations for the operators $\hat{a}$ and $\hat{b}$ are given by [9, 10]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i\left[\hat{a},\hat{H}\right] + \hat{F}_a(t), \quad (4)$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - i\left[\hat{b},\hat{H}\right] + \hat{F}_b(t), \quad (5)$$

where $\kappa$ is the cavity damping constant and $\hat{F}_a(t)$ and $\hat{F}_b(t)$ are noise operators associated with the vacuum reservoir and having the following correlation properties:

$$\left\langle\hat{F}_a(t)\right\rangle = \left\langle\hat{F}_b(t)\right\rangle = 0, \quad (6)$$

$$\left\langle\hat{F}_a^{\dagger}(t)\hat{F}_a(t')\right\rangle = \left\langle\hat{F}_b^{\dagger}(t)\hat{F}_b(t')\right\rangle = 0, \quad (7)$$

$$\langle\hat{F}_a(t)\hat{F}_a^{\dagger}(t')\rangle = \langle\hat{F}_b(t)\hat{F}_b^{\dagger}(t')\rangle = \kappa\delta(t-t'), \quad (8)$$

$$\langle\hat{F}_a^{\dagger}(t)\hat{F}_a^{\dagger}(t')\rangle = \langle\hat{F}_b^{\dagger}(t)\hat{F}_b^{\dagger}(t')\rangle = \langle\hat{F}_a(t)\hat{F}_a(t')\rangle = \langle\hat{F}_b(t)\hat{F}_b(t')\rangle = 0. \quad (9)$$

With the aid of Eqs. [1], [4], and [5], one can easily establish that

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a^k + \hat{F}_a(t), \quad (10)$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - g\hat{\sigma}_b^k + \hat{F}_b(t). \quad (11)$$

Furthermore, the master equation for a three-level atom interacting with a vacuum reservoir is given by [10]

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H},\hat{\rho}] + \frac{\gamma}{2}\begin{bmatrix} 2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\hat{\rho} - \\ \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k + 2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} \\ -\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k \end{bmatrix}, \quad (12)$$

where $\gamma$, considered to be the same for levels $|a\rangle$and $|b\rangle$, is the spontaneous emission decay constant. We can rewrite Eq. [12] as

$$\frac{d\hat{\rho}}{dt} = -i\left[\hat{H},\hat{\rho}\right] + \frac{\gamma}{2}\left[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\eta}_a^k\hat{\rho} - \hat{\rho}\hat{\eta}_a^k + 2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\eta}_b^k\hat{\rho} - \hat{\rho}\hat{\eta}_b^k\right], \quad (13)$$

where

$$\hat{\eta}_a^k = |a\rangle_{kk}\langle a|, \quad (14)$$

$$\hat{\eta}_b^k = |b\rangle_{kk}\langle b|. \quad (15)$$

Using Eq. [1], we can put Eq. [13] in the form

$$\frac{d\hat{\rho}}{dt} = g\begin{bmatrix} \hat{\sigma}_a^{\dagger k}\hat{a}\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a} + \hat{\sigma}_b^{\dagger k}\hat{b}\hat{\rho} - \\ \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{b} - \hat{a}^{\dagger}\hat{\sigma}_a^k\hat{\rho} - \hat{b}^{\dagger}\hat{\sigma}_b^k\hat{\rho} + \\ \hat{\rho}\hat{a}^{\dagger}\hat{\sigma}_a^k + \hat{\rho}\hat{b}^{\dagger}\hat{\sigma}_b^k \end{bmatrix} + \frac{\gamma}{2}\begin{bmatrix} 2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\eta}_a^k\hat{\rho} - \hat{\rho}\hat{\eta}_a^k + \\ 2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\eta}_b^k\hat{\rho} - \hat{\rho}\hat{\eta}_b^k \end{bmatrix}. \quad (16)$$

Now applying the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{A}) \quad (17)$$

along with Eq. [16], we can easily establish that

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -\gamma\langle\hat{\sigma}_a^k\rangle + g\left[\langle\hat{\eta}_b^k\hat{a}\rangle - \langle\hat{\eta}_a^k\hat{a}\rangle + \langle\hat{b}^{\dagger}\hat{\sigma}_c^k\rangle\right], \quad (18)$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = -\frac{\gamma}{2}\langle\hat{\sigma}_b^k\rangle + g\left[\langle\hat{\eta}_c^k\hat{b}\rangle - \langle\hat{\eta}_b^k\hat{b}\rangle + \langle\hat{a}^{\dagger}\hat{\sigma}_c^k\rangle\right], \quad (19)$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = -\frac{\gamma}{2}\langle\hat{\sigma}_c^k\rangle + g\left[\langle\hat{\sigma}_b^k\hat{a}\rangle - \langle\hat{\sigma}_a^k\hat{b}\rangle\right], \quad (20)$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -\gamma\langle\hat{\eta}_a^k\rangle + g\left[\left\langle\hat{\sigma}_a^{\dagger k}\hat{a}\right\rangle + \langle\hat{a}^{\dagger}\hat{\sigma}_a^k\rangle\right], \quad (21)$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = \gamma\left[\langle\hat{\eta}_a^k\rangle - \langle\hat{\eta}_b^k\rangle\right] + g\left[\langle\hat{b}^{\dagger}\hat{\sigma}_b^k\rangle + \langle\hat{\sigma}_b^{\dagger k}\hat{b}\rangle - \langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle - \langle\hat{a}^{\dagger}\hat{\sigma}_a^k\rangle\right], \quad (22)$$

$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = \gamma\langle\hat{\eta}_b^k\rangle - g\left[\langle\hat{b}^{\dagger}\hat{\sigma}_b^k\rangle + \langle\hat{\sigma}_b^{\dagger k}\hat{b}\rangle\right], \quad (23)$$

where

$$\hat{\sigma}_c^k = |c\rangle_{kk}\langle a|, \quad (24)$$

and

$$\hat{\eta}_c^k = |c\rangle_{kk}\langle c|. \quad (25)$$

We see that Eqs. [18]-[23] are nonlinear differential equations and hence it is not possible to find exact time-dependent solutions of these equations. We intend to overcome this problem by applying the large-time approximation [13]. Then using this approximation scheme, we get from Eqs. [10] and [11] the approximately valid relations

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a^k + \frac{2}{\kappa}\hat{F}_a(t), \quad (26)$$

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b^k + \frac{2}{\kappa}\hat{F}_b(t). \quad (27)$$

Evidently, these would turn out to be exact relations at steady state. Now combining Eqs. [26] and [27] with Eqs. [18]-[23], we get

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -[\gamma+\gamma_c]\langle\hat{\sigma}_a^k\rangle + \frac{2g}{\kappa}\left[\langle\hat{\eta}_b^k\hat{F}_a(t)\rangle - \langle\hat{\eta}_a^k\hat{F}_a(t)\rangle + \langle\hat{F}_b^\dagger(t)\hat{\sigma}_c^k\rangle\right], \quad (28)$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = -\left[\frac{\gamma}{2}+\frac{\gamma_c}{2}\right]\langle\hat{\sigma}_b^k\rangle + \frac{2g}{\kappa}\left[\langle\hat{\eta}_c^k\hat{F}_b(t)\rangle - \langle\hat{\eta}_b^k\hat{F}_b(t)\rangle - \langle\hat{F}_a^\dagger(t)\hat{\sigma}_c^k\rangle\right], \quad (29)$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = -\left[\frac{\gamma}{2}+\frac{\gamma_c}{2}\right]\langle\hat{\sigma}_c^k\rangle + \frac{2g}{\kappa}\left[\langle\hat{\sigma}_b^k\hat{F}_a(t)\rangle - \langle\hat{\sigma}_a^k\hat{F}_b(t)\rangle\right], \quad (30)$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -[\gamma+\gamma_c]\langle\hat{\eta}_a^k\rangle + \frac{2g}{\kappa}\left[\langle\hat{\sigma}_a^{\dagger k}\hat{F}_a(t)\rangle + \langle\hat{F}_a^\dagger(t)\hat{\sigma}_a^k\rangle\right], \quad (31)$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = -[\gamma+\gamma_c]\langle\hat{\eta}_b^k\rangle + [\gamma+\gamma_c]\langle\hat{\eta}_a^k\rangle + \frac{2g}{\kappa}\left[\begin{matrix}\langle\hat{F}_b^\dagger(t)\hat{\sigma}_b^k\rangle + \langle\hat{\sigma}_b^{\dagger k}\hat{F}_b(t)\rangle - \\ \langle\hat{\sigma}_a^{\dagger k}\hat{F}_a(t)\rangle - \langle\hat{F}_a^\dagger(t)\hat{\sigma}_a^k\rangle\end{matrix}\right], \quad (32)$$

$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = [\gamma+\gamma_c]\langle\hat{\eta}_b^k\rangle - \frac{2g}{\kappa}\left[\langle\hat{F}_b^\dagger(t)\hat{\sigma}_b^k\rangle + \langle\hat{\sigma}_b^{\dagger k}\hat{F}_b(t)\rangle\right], \quad (33)$$

where

$$\gamma_c = \frac{4g^2}{\kappa} \quad (34)$$

is the stimulated emission decay constant.

We next proceed to find the expectation value of the product involving a noise operator and an atomic operator that appears in Eqs. [28] - [33]. To this end, after removing the angular brackets, Eq. [31] can be rewritten as

$$\frac{d}{dt}\hat{\eta}_a^k = -[\gamma+\gamma_c]\hat{\eta}_a^k + \frac{2g}{\kappa}\left[\hat{\sigma}_a^{\dagger k}\hat{F}_a(t) + \hat{F}_a^\dagger(t)\hat{\sigma}_a^k\right] + \hat{f}_a(t), \quad (35)$$

where $\hat{f}_a(t)$ is the noise operator associated with $\hat{\eta}_a$. A formal solution of this equation can be written as

$$\hat{\eta}_a^k(t) = \hat{\eta}_a^k(0)e^{-(\gamma+\gamma_c)t} + \int_0^t e^{-(\gamma+\gamma_c)(t-t')}\left[\frac{2g}{\kappa}\left[\hat{\sigma}_a^{\dagger k}(t')\hat{F}_a(t') + \hat{F}_a^\dagger(t')\hat{\sigma}_a^k(t')\right] + \hat{f}_a(t')\right]dt'. \quad (36)$$

Multiplying Eq. [36] on the right by $\hat{F}_a(t)$ and taking the expectation value of the resulting equation, we have

$$\langle\hat{\eta}_a^k(t)\hat{F}_a(t)\rangle = \langle\hat{\eta}_a^k(0)\hat{F}_a(t)\rangle e^{-(\gamma+\gamma_c)t} + \int_0^t e^{-(\gamma+\gamma_c)(t-t')}\left[\frac{2g}{\kappa}\left[\begin{matrix}\langle\hat{\sigma}_a^{\dagger k}(t')\hat{F}_a(t')\hat{F}_a(t)\rangle + \\ \langle\hat{F}_a^\dagger(t')\hat{\sigma}_a^k(t')\hat{F}_a(t)\rangle\end{matrix}\right] + \langle\hat{f}_a(t')\hat{F}_a(t)\rangle\right]dt'. \quad (37)$$

Ignoring the noncommutativity of the atomic and noise operators and neglecting the correlation between $\hat{F}_a(t)$ and $\hat{\sigma}_a^k(t')$, assumed to be considerably small [6] , one can write the approximately valid relations

$$\langle\hat{\sigma}_a^{\dagger k}(t')\hat{F}_a(t')\hat{F}_a(t)\rangle = \langle\hat{\sigma}_a^{\dagger k}(t')\rangle\langle\hat{F}_a(t')\hat{F}_a(t)\rangle = 0, \quad (38)$$

$$\langle\hat{F}_a^\dagger(t')\hat{\sigma}_a^k(t')\hat{F}_a(t)\rangle = \langle\hat{\sigma}_a^k(t')\rangle\langle\hat{F}_a^\dagger(t')\hat{F}_a(t)\rangle = 0, (1) \quad (39)$$

$$\langle\hat{f}_a(t')\hat{F}_a(t)\rangle = \langle\hat{f}_a(t')\rangle\langle\hat{F}_a(t)\rangle = 0. \quad (40)$$

Now on account of these approximately valid relations along with the fact that a noise operator $\hat{F}$ at a certain time should not affect the atomic variable at earlier time, Eq. [37] takes the form

$$\langle\hat{\eta}_a^k(t)\hat{F}_a(t)\rangle = 0. \quad (41)$$

Following a similar procedure, one can also check that

$$\langle\hat{\eta}_b^k(t)\hat{F}_a(t)\rangle = 0, \quad (42)$$

$$\langle\hat{\eta}_c^k(t)\hat{F}_b(t)\rangle = 0, \quad (43)$$

$$\langle\hat{\eta}_b^k(t)\hat{F}_b(t)\rangle = 0, \quad (44)$$

$$\langle\hat{F}_a^\dagger(t)\hat{\sigma}_a^k(t)\rangle = 0, \quad (45)$$

$$\langle\hat{F}_b^\dagger(t)\hat{\sigma}_b^k(t)\rangle = 0. \quad (46)$$

We also take

$$\langle\hat{F}_a^\dagger(t)\hat{\sigma}_c^k(t)\rangle = \langle\hat{F}_b^\dagger(t)\hat{\sigma}_c^k(t)\rangle = 0. \quad (47)$$

With the aid of Eqs. [41]-[47], we rewrite Eqs. [28], [29], [31], [32], and [33] as

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -[\gamma+\gamma_c]\langle\hat{\sigma}_a^k\rangle, \quad (48)$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = -\left[\frac{\gamma}{2}+\frac{\gamma_c}{2}\right]\langle\hat{\sigma}_b^k\rangle. \quad (49)$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -[\gamma+\gamma_c]\langle\hat{\eta}_a^k\rangle, \quad (50)$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = -[\gamma+\gamma_c]\langle\hat{\eta}_b^k\rangle + [\gamma+\gamma_c]\langle\hat{\eta}_a^k\rangle, \quad (51)$$

$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = [\gamma+\gamma_c]\langle\hat{\eta}_b^k\rangle. \quad (52)$$

We note that Eqs. [48] - [52] represent the equation of evolution for the atomic operators in the absence of the pumping process. The pumping process must surely affect the dynamics of $\langle\hat{\eta}_a^k\rangle$ and $\langle\hat{\eta}_c^k\rangle$. We seek here to pump the atoms by electron bombardment. If $r_a$ represents the rate at which a single atom is pumped from the bottom to the top level, then $\langle\hat{\eta}_a^k\rangle$ increases at the rate of $r_a\langle\hat{\eta}_c^k\rangle$ and $\langle\hat{\eta}_c^k\rangle$ decreases at the same rate. In view of this, we rewrite Eqs. [50] and [52] as

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -[\gamma+\gamma_c]\langle\hat{\eta}_a^k\rangle + r_a\langle\hat{\eta}_c^k\rangle, \quad (53)$$

$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = [\gamma+\gamma_c]\langle\hat{\eta}_b^k\rangle - r_a\langle\hat{\eta}_c\rangle. \quad (54)$$

We next sum Eqs. [48], [49], [51], [53], and [54] over the N three-level atoms, so that

$$\frac{d}{dt}\langle\hat{m}_a\rangle = -[\gamma+\gamma_c]\langle\hat{m}_a\rangle, \quad (55)$$

$$\frac{d}{dt}\langle\hat{m}_b\rangle = -\left[\frac{\gamma}{2}+\frac{\gamma_c}{2}\right]\langle\hat{m}_b\rangle, \quad (56)$$

$$\frac{d}{dt}\langle\hat{N}_a\rangle = -[\gamma+\gamma_c]\langle\hat{N}_a\rangle + r_a\langle\hat{N}_c\rangle, \quad (57)$$

$$\frac{d}{dt}\langle\hat{N}_b\rangle = -[\gamma+\gamma_c]\langle\hat{N}_b\rangle + [\gamma+\gamma_c]\langle\hat{N}_a\rangle, \quad (58)$$

$$\frac{d}{dt}\langle\hat{N}_c\rangle = [\gamma+\gamma_c]\langle\hat{N}_b\rangle - r_a\langle\hat{N}_c\rangle, \quad (59)$$

in which

$$\hat{m}_a = \sum_{k=1}^N \hat{\sigma}_a^k, \quad (60)$$

$$\hat{m}_b = \sum_{k=1}^N \hat{\sigma}_b^k, \quad (61)$$

$$\hat{N}_a = \sum_{k=1}^N \hat{\eta}_a^k, \quad (62)$$

$$\hat{N}_b = \sum_{k=1}^N \hat{\eta}_b^k, \quad (63)$$

$$\hat{N}_c = \sum_{k=1}^N \hat{\eta}_c^k, \quad (64)$$

with the operators $\hat{N}_a$, $\hat{N}_b$, and $\hat{N}_c$ representing the number of atoms in the top, intermediate, and bottom levels. In addition, employing the completeness relation

$$\hat{\eta}_a^k + \hat{\eta}_b^k + \hat{\eta}_c^k = \hat{I}, \quad (65)$$

we easily arrive at

$$\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle + \langle\hat{N}_c\rangle = N. \quad (66)$$

Furthermore, applying the definition given by Eq. [2] and setting for any k

$$\hat{\sigma}_a^k = |b\rangle\langle a|, \tag{67}$$

we have

$$\hat{m}_a = N|b\rangle\langle a|. \tag{68}$$

Following the same procedure, one can also check that

$$\hat{m}_b = N|c\rangle\langle b|, \tag{69}$$

$$\hat{m}_c = N|c\rangle\langle a|, \tag{70}$$

$$\hat{N}_a = N|a\rangle\langle a|, \tag{71}$$

$$\hat{N}_b = N|b\rangle\langle b|, \tag{72}$$

$$\hat{N}_c = N|c\rangle\langle c|, \tag{73}$$

where

$$\hat{m}_c = \sum_{k=1}^{N} \hat{\sigma}_c^k. \tag{74}$$

Moreover, using the definition

$$\hat{m} = \hat{m}_a + \hat{m}_b \tag{75}$$

and taking into account Eqs. [68]-[73], it can be readily established that

$$\hat{m}^\dagger \hat{m} = N\big(\hat{N}_a + \hat{N}_b\big), \tag{76}$$

$$\hat{m}\hat{m}^\dagger = N\big(\hat{N}_b + \hat{N}_c\big), \tag{77}$$

$$\hat{m}^2 = N\hat{m}_c. \tag{78}$$

With the aid of Eq. [66], one can put Eq. [57] in the form

$$\frac{d}{dt}\langle \hat{N}_a\rangle = -[\gamma + \gamma_c + r_a]\langle \hat{N}_a\rangle + r_a[N - \langle \hat{N}_b\rangle]. \tag{79}$$

Applying the large-time approximation scheme to Eq. [58], we get

$$\langle \hat{N}_b\rangle = \langle \hat{N}_a\rangle. \tag{80}$$

Thus on taking into account this result, Eq. [79] can be written as

$$\frac{d}{dt}\langle \hat{N}_a\rangle = -[\gamma + \gamma_c + 2r_a]\langle \hat{N}_a\rangle + Nr_a. \tag{81}$$

The steady-state solution of Eq. [81] is expressible as

$$\langle \hat{N}_a\rangle = \frac{r_a N}{\gamma + \gamma_c + 2r_a}. \tag{82}$$

Using the steady-state solution of Eq. [59] along with Eq. [80], we have

$$\langle \hat{N}_c\rangle = \frac{\gamma + \gamma_c}{r_a}\langle \hat{N}_a\rangle. \tag{83}$$

On account of Eq. [82], Eq. [83] takes the form

$$\langle \hat{N}_c\rangle = \frac{(\gamma + \gamma_c)N}{\gamma + \gamma_c + 2r_a}. \tag{84}$$

For $r_a = 0$, we see that $\langle \hat{N}_a\rangle = \langle \hat{N}_b\rangle = 0$ and $\langle \hat{N}_c\rangle = N$. This result holds whether the atoms are initially in the top or bottom level.

In the presence of N three-level atoms, we rewrite Eq. [10] as [10]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \lambda \hat{m}_a + \beta \hat{F}_a(t), \tag{85}$$

in which $\lambda$ and $\beta$ are constants whose values remain to be fixed. Applying Eq. [26], we get

$$[\hat{a}, \hat{a}^\dagger]_k = \frac{4g^2}{\kappa^2}(\hat{\eta}_b^k - \hat{\eta}_a^k) + \frac{4}{\kappa^2}[F_a, F_a^\dagger] \tag{86}$$

and on summing over all atoms, we have

$$[\hat{a}, \hat{a}^\dagger] = \frac{4g^2}{\kappa^2}\big(\hat{N}_b - \hat{N}_a\big) + \frac{4N}{\kappa^2}[F_a, F_a^\dagger], \tag{87}$$

where

$$[\hat{a}, \hat{a}^\dagger] = \sum_{k=1}^{N}[\hat{a}, \hat{a}^\dagger]_k \tag{88}$$

stands for the commutator of $\hat{a}$ and $\hat{a}^\dagger$ when light mode $a$ is interecting with all the N three-level atoms. On the other hand, applying the large-time approximation to Eq. [85], one can easily find

$$[\hat{a}, \hat{a}^\dagger] = N\frac{4\lambda^2}{\kappa^2}\big(\hat{N}_b - \hat{N}_a\big) + \frac{4\beta^2}{\kappa^2}[F_a, F_a^\dagger]. \tag{89}$$

Thus on account of Eqs. [87] and [89], we see that

$$\lambda = \pm\frac{g}{\sqrt{N}}, \tag{90}$$

$$\beta = \pm\sqrt{N}. \tag{91}$$

In view of Eqs. [90] and [91], Eq. [85] can be written as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \frac{g}{\sqrt{N}}\hat{m}_a + \sqrt{N}\hat{F}_a(t). \tag{92}$$

Following a similar procedure, one can also readily establish that

$$[\hat{b}, \hat{b}^\dagger] = \frac{4g^2}{\kappa^2}\big(\hat{N}_c - \hat{N}_b\big) + \frac{4N}{\kappa^2}[F_b, F_b^\dagger], \tag{93}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \frac{g}{\sqrt{N}}\hat{m}_b + \sqrt{N}\hat{F}_b(t). \tag{94}$$

Furthermore, in order to include the effect of pumping process, we rewrite Eqs. [55] and [56] as

$$\frac{d}{dt}\hat{m}_a = -\frac{\mu}{2}\hat{m}_a + \hat{G}_a(t), \tag{95}$$

$$\frac{d}{dt}\hat{m}_b = -\frac{\mu}{2}\hat{m}_b + \hat{G}_b(t) \tag{96}$$

in which $\hat{G}_a(t)$ and $\hat{G}_b(t)$ are noise operators with vanishing mean and $\mu$ is a parameter whose value remains to be determined. Employing the relation

$$\frac{d}{dt}\langle \hat{m}_a^\dagger \hat{m}_a\rangle = \left\langle \frac{d\hat{m}_a^\dagger}{dt}\hat{m}_a\right\rangle + \left\langle \hat{m}_a^\dagger \frac{d\hat{m}_a}{dt}\right\rangle \tag{97}$$

along with Eq. [95], we easily find

$$\frac{d}{dt}\langle \hat{m}_a^\dagger \hat{m}_a\rangle = -\mu\langle \hat{m}_a^\dagger \hat{m}_a\rangle + \langle \hat{m}_a^\dagger \hat{G}_a(t)\rangle + \langle \hat{G}_a^\dagger(t)\hat{m}_a\rangle, \tag{98}$$

from which follows

$$\frac{d}{dt}\langle \hat{N}_a\rangle = -\mu\langle \hat{N}_a\rangle + \frac{1}{N}\big[\langle \hat{m}_a^\dagger \hat{G}_a(t)\rangle + \langle \hat{G}_a^\dagger(t)\hat{m}_a\rangle\big]. \tag{99}$$

Now comparison of Eqs. [81] and [99] shows that

$$\mu = \gamma + \gamma_c + 2r_a \tag{100}$$

and

$$\langle \hat{m}_a^\dagger \hat{G}_a(t)\rangle + \langle \hat{G}_a^\dagger(t)\hat{m}_a\rangle = r_a N^2. \tag{101}$$

We observe that Eq. [101] is equivalent to

$$\langle \hat{G}_a^\dagger(t)\hat{G}_a(t')\rangle = r_a N^2\delta(t - t'). \tag{102}$$

One can also easily verify that

$$\langle \hat{G}_b(t)\hat{G}_b^\dagger(t')\rangle = (\gamma + \gamma_c)N^2\delta(t - t'). \tag{103}$$

Furthermore, adding Eqs. [55] and [56], we have

$$\frac{d}{dt}\langle \hat{m}\rangle = -\frac{1}{2}[\gamma + \gamma_c]\langle \hat{m}\rangle - \frac{1}{2}[\gamma + \gamma_c]\langle \hat{m}_a\rangle, \tag{104}$$

where $\hat{m}$ is given by Eq. [75]. Upon casting Eq. [104] into the form

$$\frac{d}{dt}\hat{m} = -\frac{\mu}{2}\hat{m} - \frac{\mu}{2}\hat{m}_a + \hat{G}(t), \tag{105}$$

one can also easily verify that $\mu$ has the value given by Eq. [100] and

$$\langle \hat{G}^\dagger(t)\hat{G}(t')\rangle = r_a N^2\delta(t - t'). \tag{106}$$

On the other hand, assuming the atoms to be initial in the bottom level, the expectation value of the solution of Eq. [95] happens to be

$$\langle \hat{m}_a(t)\rangle = 0. \tag{107}$$

Hence the expectation value of the solution of Eq. [92] turns out to be

$$\langle \hat{a}(t)\rangle = 0. \tag{108}$$

In view of Eqs. [92] and [108], we claim that $\hat{a}(t)$ is a Gaussian variable with zero mean. One can also easily verify that

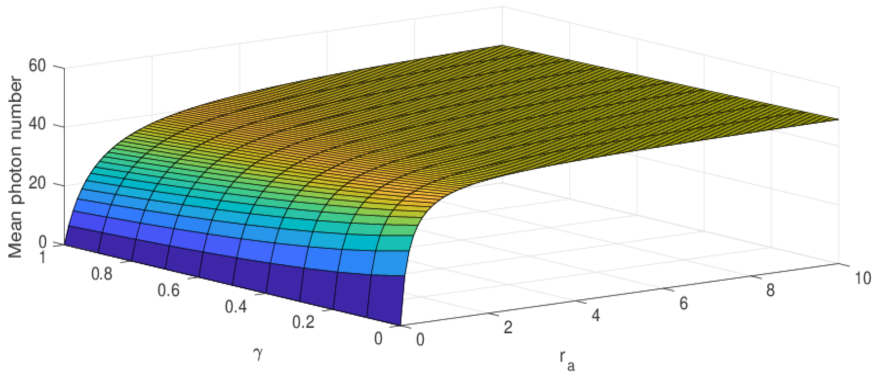
$$\langle \hat{b}(t)\rangle = 0. \tag{109}$$

Then on account of Eqs. [94] and [109], we realize that $\hat{b}(t)$ is a Gaussian variable with zero mean.

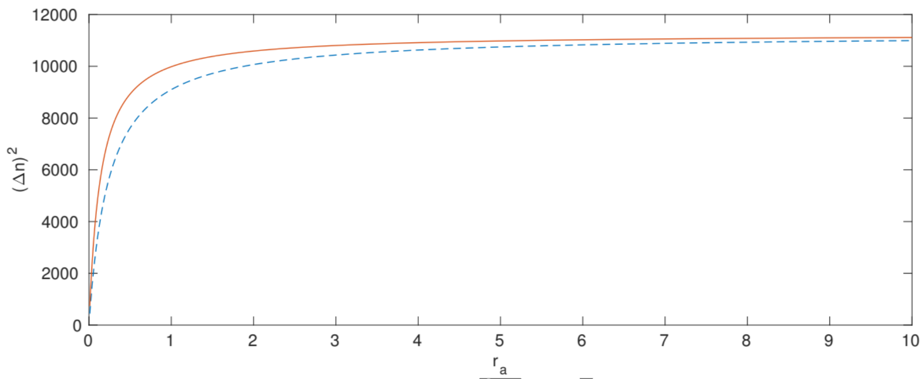
Furthermore adding Eqs. [108] and [109], we obtain

$$\langle \hat{c}\rangle = 0 \tag{110}$$

where



Φιγυρε 3:Plots of the mean photon number for the two-mode cavity light at steady state [Eq. [133]] vs $r_a$ and $\gamma$ for $\kappa = 0.8$, $\gamma_c = 0.4$, and $N = 100$.



Φιγυρε 4:Plots of the photon number variance of two-mode cavity light at steady state, [Eq. [151]] for $\kappa = 0.8$, $\gamma_c = 0.4$, $\gamma = 0$(solid curve), $\gamma = 0.2$(dashed curve), and $N = 100$.

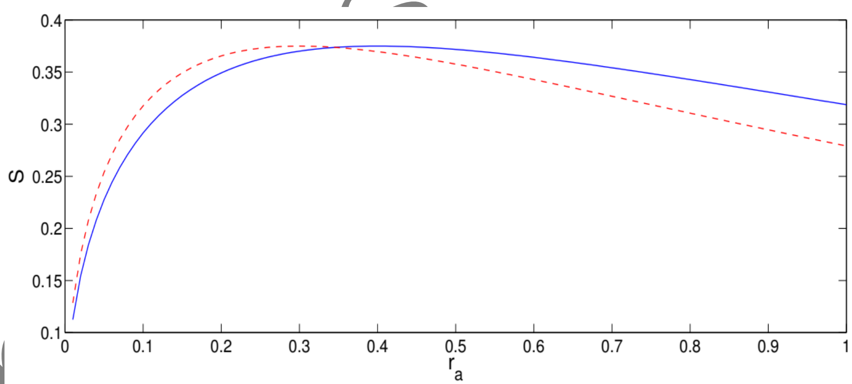
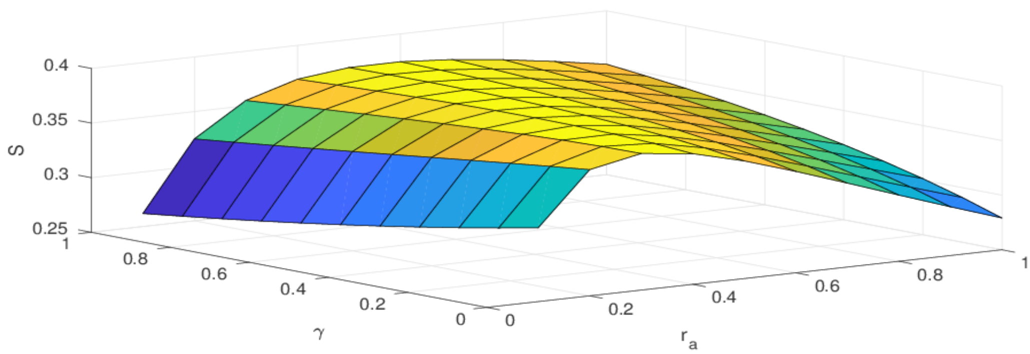


Figure5: Plots of the quadrature squeezing at steady state, [Eq. [165]] versus $r_a$ for $\kappa = 0.2$, $\gamma_c = 1.2$, $\gamma = 0$ (dashed curve), and for $\gamma = 0.4$ (solid curve).



Φιγυρε 6:Plots of the quadrature squeezing at steady state, [Eq. [165]] versus $r_a$ and $\gamma$ for $\kappa = 0.2$, $\gamma_c = 1.2$.

$$\hat{c} = \hat{a} + \hat{b}. \tag{111}$$

In addition, adding Eqs.[92] and [94], we get

$$\frac{d\hat{c}}{dt} = -\frac{\kappa}{2}\hat{c} + \frac{g}{\sqrt{N}}\hat{m} + \sqrt{N}\hat{F}_c(t), \tag{112}$$

where

$$\hat{F}_c(t) = \hat{F}_a(t) + \hat{F}_b(t) \tag{113}$$

and $\hat{m}$ is given by Eq. [75]. One can also easily check that

$$\langle \hat{F}_c(t) \rangle = 0, \tag{114}$$

$$\langle \hat{F}_c^\dagger(t)\hat{F}_c(t') \rangle = 0, \tag{115}$$

$$\langle \hat{F}_c^\dagger(t)\hat{F}_c^\dagger(t') \rangle = \langle \hat{F}_c(t)\hat{F}_c(t) \rangle = 0, \tag{116}$$

$$\langle \hat{F}_c(t)\hat{F}_c^\dagger(t') \rangle = 2\kappa\delta(t-t'). \tag{117}$$

In view of Eqs. [110] and [112], we see that $\hat{c}$ is a Gaussian variable with zero mean.

## 3. Photon statistics

In this section we wish to calculate the mean and variance of the photon number for the two-mode cavity light at steady state. To this end, using the relation

$$\frac{d}{dt}\langle \hat{c}^\dagger(t)\hat{c}(t) \rangle = \left\langle \frac{d\hat{c}^\dagger(t)}{dt}\hat{c}(t) \right\rangle + \left\langle \hat{c}^\dagger(t)\frac{d\hat{c}(t)}{dt} \right\rangle \tag{118}$$

along with Eq. [112], we readily find

$$\frac{d}{dt}\langle \hat{c}^\dagger t)\hat{c}(t) \rangle = -\kappa\langle \hat{c}^\dagger(t)\hat{c}(t) \rangle + \frac{g}{\sqrt{N}}[\langle \hat{c}^\dagger(t)\hat{m}(t) \rangle + \langle \hat{m}^\dagger(t)\hat{c}(t) \rangle]$$
$$+\sqrt{N}[\langle \hat{F}_c^\dagger(t)\hat{c}(t) \rangle + \langle \hat{c}^\dagger(t)\hat{F}_c(t) \rangle].$$

(119)

Next we seek to evaluate $\langle \hat{c}^\dagger(t)\hat{m}(t) \rangle$. Applying the large-time approximation, one gets from Eq. [112] the approximately valid relation

$$\hat{c}(t) = \frac{2g}{\kappa\sqrt{N}}\hat{m} + \frac{2\sqrt{N}}{\kappa}\hat{F}_c(t). \tag{120}$$

Multiplying the adjoint of Eq. [120] on the right by $\hat{m}(t)$ and taking the expectation value of the resulting expression, we get

$$\langle \hat{c}^\dagger(t)\hat{m}(t) \rangle = \frac{2g\sqrt{N}}{\kappa}[\langle \hat{N}_a(t) \rangle + \langle \hat{N}_b(t) \rangle] + \frac{2\sqrt{N}}{\kappa}\langle \hat{F}_c^\dagger(t)\hat{m}(t) \rangle. \tag{121}$$

We now proceed to evaluate $\langle \hat{F}_c^\dagger(t)\hat{m}(t) \rangle$. To this end, a formal solution of Eq. [105] can be written as

$$\hat{m}(t) = \hat{m}(0)e^{-\frac{\mu}{2}t} + \int_0^t e^{-\frac{\mu}{2}(t-t\prime)}\left[-\frac{\mu}{2}\hat{m}_a(t') + \hat{G}(t')\right]dt'. \tag{122}$$

Multiplying Eq. [122] on the left by $\hat{F}_c^\dagger(t)$ and taking the expectation value of the resulting expression, we have

$$\langle \hat{F}_c^\dagger(t)\hat{m}(t) \rangle = \langle \hat{F}_c^\dagger(t)\hat{m}(0) \rangle e^{-\frac{\mu}{2}t} + \int_0^t e^{-\frac{\mu}{2}(t-t')}\left[-\frac{\mu}{2}\langle \hat{F}_c^\dagger(t)\hat{m}_a(t') \rangle + \langle \hat{F}_c^\dagger(t)\hat{G}(t') \rangle\right]dt'. \tag{123}$$

Taking into account the fact that a noise operator $\hat{F}$ at a certain time should not affect the atomic variable at earlier time and assuming that the cavity mode and atomic mode operators are not correlated, we get

$$\langle \hat{F}_c^\dagger(t)\hat{m}(t) \rangle = 0. \tag{124}$$

On account of this result, Eq. [121] takes the form

$$\langle \hat{c}^\dagger(t)\hat{m}(t) \rangle = \frac{2g\sqrt{N}}{\kappa}[\langle \hat{N}_a(t) \rangle + \langle \hat{N}_b(t) \rangle]. \tag{125}$$

We next seek to evaluate $\langle \hat{F}_c^\dagger(t)\hat{c}(t) \rangle$. To this end, a formal solution of Eq. [112] can be written as

$$\hat{c}(t) = \hat{c}(0)e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\left[\frac{g}{\sqrt{N}}\hat{m}(t') + \sqrt{N}\hat{F}_c(t')\right]dt'. \tag{126}$$

Multiplying Eq. [126] on the left by $\hat{F}_c^\dagger(t)$ and taking the expectation value of the resulting expression, we get

$$\langle \hat{F}_c^\dagger(t)\hat{c}(t) \rangle = \langle \hat{F}_c^\dagger(t)\hat{c}(0) \rangle e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\left[\frac{g}{\sqrt{N}}\langle \hat{F}_c^\dagger(t)\hat{m}(t') \rangle + \sqrt{N}\langle \hat{F}_c^\dagger(t)\hat{F}_c(t') \rangle\right]dt'. \tag{127}$$

In view of Eqs. [115] and [124] along with the fact that a noise operator $\hat{F}$ at a certain time should not affect the atomic variable at earlier time, Eq. [127] becomes

$$\langle \hat{F}_c^\dagger(t)\hat{c}(t) \rangle = 0. \tag{128}$$

Now on account of Eqs. [125] and [128] along with their complex conjugates, we can rewrite Eq. [119] as □□□□□□□□□□Plots of the mean photon number for the two-mode cavity light at steady state [Eq. [133]] for $\kappa = 0.8$, $\gamma_c = 0.4$, $\gamma = 0.2$ (dashed curve), $\gamma = 0$ (solid curve), and $N = 100$.

$$\frac{d}{dt}\langle \hat{c}^\dagger(t)\hat{c}(t) \rangle = -\kappa\langle \hat{c}^\dagger(t)\hat{c}(t) \rangle + \frac{4g^2}{\kappa}[\langle \hat{N}_a(t) \rangle + \langle \hat{N}_b(t) \rangle]. \tag{129}$$

The steady-state solution of this equation is expressible as

$$\langle \hat{c}^\dagger\hat{c} \rangle = \frac{\gamma_c}{\kappa}[\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle]. \tag{130}$$

Following a similar procedure, one can establish that

$$\langle \hat{c}\hat{c}^\dagger \rangle = \frac{\gamma_c}{\kappa}[\langle \hat{N}_c \rangle + \langle \hat{N}_b \rangle] + 2N, \tag{131}$$

$$\langle \hat{c}^2 \rangle = \frac{\gamma_c}{\kappa}\langle \hat{m}_c \rangle. \tag{132}$$

In view of Eqs. [80], [82], and [84], Eqs. [130] and [131] can be rewritten as

$$\langle \hat{c}^\dagger\hat{c} \rangle = \frac{\gamma_c}{\kappa}\left[\frac{2Nr_a}{\gamma+\gamma_c+2r_a}\right], \tag{133}$$

$$\langle \hat{c}\hat{c}^\dagger \rangle = \frac{\gamma_c}{\kappa}\left[\frac{\gamma+\gamma_c+r_a}{\gamma+\gamma_c+2r_a}\right]N + 2N. \tag{134}$$

In the absence of spontaneous emission ($\gamma = 0$), the mean photon number for the two-mode cavity light has the form

$$□ = \frac{\gamma_c}{\kappa}\left(\frac{2Nr_a}{\gamma_c+2r_a}\right). \tag{135}$$

It can be seen from the plots in Fig. 2 and 3 that the presence of spontaneous emission leads to a decrease in the mean photon number for the two-mode cavity light.

Furthermore, the variance of the photon number for the two-mode cavity light is expressible as

$$(\Delta n)^2 = \langle (\hat{c}^\dagger\hat{c})^2 \rangle - \langle \hat{c}^\dagger\hat{c} \rangle^2. \tag{136}$$

Using the fact that $\hat{c}$ is a Gaussian variable with zero mean, we readily get

$$(\Delta n)^2 = \langle \hat{c}^\dagger\hat{c} \rangle\langle \hat{c}\hat{c}^\dagger \rangle + \langle \hat{c}^{2\dagger} \rangle\langle \hat{c}^2 \rangle. \tag{137}$$

We now proceed to calculate the expectation value of the atomic operator $\hat{m}_c$ following the approach presented in [10]. To this end, applying the identity given by Eq. [65], the state vector of a three-level atom can be put in the form

$$|\psi_k\rangle = c_a|a_k\rangle + c_b|b_k\rangle + c_c|c_k\rangle, \tag{138}$$

in which

$$c_a = \langle a_k|\psi_k\rangle, \tag{139}$$

$$c_b = \langle b_k|\psi_k\rangle, \tag{140}$$

$$c_c = \langle c_k|\psi_k\rangle. \tag{141}$$

The state vector described by Eq. [138] can be used to determine the expectation value of an atomic operator formed by a pair of identical energy levels or by two distinct energy levels between which transition with the emission of a photon is dipole forbidden. One can thus readily establish that

$$\langle \hat{\eta}_a^k \rangle = c_a c_a^*, \tag{142}$$

$$\langle \hat{\eta}_c^k \rangle = c_c c_c^*, \tag{143}$$

and

$$\langle \hat{\sigma}_c^k \rangle = c_a c_c^*. \tag{144}$$

We then see that

$$|\langle \hat{\sigma}_c^k \rangle|^2 = \langle \hat{\eta}_a^k \rangle \langle \hat{\eta}_c^k \rangle, \tag{145}$$

and on taking $|\langle \hat{\sigma}_c^k \rangle|$ to be real, we see that

$$|\langle \hat{\sigma}_c^k \rangle| = \sqrt{\langle \hat{\eta}_a^k \rangle \langle \hat{\eta}_c^k \rangle} \tag{146}$$

so that upon summing over k from 1 up to N, we get

$$\langle \hat{m}_c \rangle = \sqrt{\langle \hat{N}_a \rangle \langle \hat{N}_c \rangle}. \tag{147}$$

On account of this, Eq. [132] takes the form

$$\langle \hat{c}^2 \rangle = \frac{\gamma_c}{\kappa} \sqrt{\langle \hat{N}_a \rangle \langle \hat{N}_c \rangle}. \tag{148}$$

Now using Eq. [83], we have

$$\langle \hat{c}^2 \rangle = \frac{\gamma_c}{\kappa} \sqrt{\frac{\gamma + \gamma_c}{r_a}} \langle \hat{N}_a \rangle. \tag{149}$$

In view of Eqs. [130], [131], and [149], Eq. [137] becomes

$$\begin{pmatrix} (\Delta n)^2 = \frac{\gamma_c}{\kappa}\left(\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle\right) \\ \left(\frac{\gamma_c}{\kappa}\left(\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle\right) + 2N\right) \end{pmatrix} + \left(\frac{\gamma_c}{\kappa} \sqrt{\frac{\gamma + \gamma_c}{r_a}} \langle \hat{N}_a \rangle\right)^2. \tag{150}$$

Finally, on account of Eqs. [80], [82], [83], and [84] along with Eq. [150], we arrive at

$$(\Delta n)^2 = \frac{1}{4} ?^2 (3\eta + 2) + 2N ?, \tag{151}$$

where

$$\eta = \frac{\gamma + \gamma_c}{r_a}. \tag{152}$$

Now inspection of Eq. [151] indicates that $(\Delta n)^2 > ?$ and hence the photon statistics of the two-mode cavity light is super-Poissonian. Our result shows that the photon number variance of the two-mode cavity light is greater than the one obtained by Menisha [17]. This must be due to the reservoir noise operators. The plots in Fig. 4. indicate that the effect of spontaneous emission is to decrease the variance of the photon number.

## 4. Quadrature squeezing

We now proceed to calculate the quadrature squeezing of the two-mode cavity light in the entire frequency interval. To this end, the squeezing properties of the two-mode cavity light are described by two quadrature operators defined by

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c} \tag{153}$$

and

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}). \tag{154}$$

It can be readily established that [12]

$$[\hat{c}_-, \hat{c}_+] = 2i\frac{\gamma_c}{\kappa}\left(\langle \hat{N}_a \rangle - \langle \hat{N}_c \rangle\right) - 4Ni. \tag{155}$$

It then follows that [13]

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}\left(\langle \hat{N}_c \rangle - \langle \hat{N}_a \rangle\right) + 2N. \tag{156}$$

Upon setting $r_a = 0$, we see that

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa} N + 2N. \tag{157}$$

This represents the quadrature variance for two-mode vacuum state. The variance of the quadrature operator is expressible as

$$(\Delta c_\pm)^2 = \pm\langle (\hat{c}^\dagger \pm \hat{c})^2 \rangle \mp [\langle \hat{c}^\dagger + \hat{c} \rangle]^2, \tag{158}$$

so that on account of Eq. [110], we have

$$(\Delta c_\pm)^2 = \langle \hat{c}^\dagger \hat{c} \rangle + \langle \hat{c} \hat{c}^\dagger \rangle \pm \langle \hat{c}^{\dagger 2} \rangle \pm \langle \hat{c}^2 \rangle. \tag{159}$$

Now employing Eqs. [66], [130], [131], and [149], we arrive at

$$(\Delta c_+)^2 = \frac{\gamma_c}{\kappa}\left(N + \langle \hat{N}_a \rangle + 2\sqrt{\frac{\gamma + \gamma_c}{r_a}} \langle \hat{N}_a \rangle\right) + 2N, \tag{160}$$

$$(\Delta c_-)^2 = \frac{\gamma_c}{\kappa}\left(N + \langle \hat{N}_a \rangle - 2\sqrt{\frac{\gamma + \gamma_c}{r_a}} \langle \hat{N}_a \rangle\right) + 2N. \tag{161}$$

Moreover, on setting $r_a = 0$ in Eqs. [160] and [161], we get

$$(\Delta c_+)_v^2 = (\Delta c_-)_v^2 = \frac{\gamma_c}{\kappa} N + 2N. \tag{162}$$

This represents the quadrature variance of a two-mode cavity vacuum state. From Eqs. [157] and [162], we see that the two-mode cavity light is in a minimum uncertainty state. We seek to calculate the quadrature squeezing of the two-mode cavity light relative to the quadrature variance of the two-mode cavity vacuum state. We then define the quadrature squeezing of the two-mode cavity light by

$$S = \frac{(\Delta c_-)_v^2 - (\Delta c_-)^2}{(\Delta c_-)_v^2}. \tag{163}$$

Now employing Eqs. [161] and [162], one can put Eq. [163] in the form

$$S = \frac{\frac{\gamma_c}{\kappa}\left(2\sqrt{\frac{\gamma + \gamma_c}{r_a}} \langle \hat{N}_a \rangle - \langle \hat{N}_a \rangle\right)}{\frac{\gamma_c}{\kappa} N + 2N}. \tag{164}$$

On account of Eq. [82], Eq. [164] takes the form

$$S = \frac{\gamma_c}{\gamma_c + 2\kappa}\left(\frac{2\sqrt{\frac{\gamma + \gamma_c}{r_a}} - 1}{\frac{\gamma + \gamma_c}{r_a} + 2}\right). \tag{165}$$

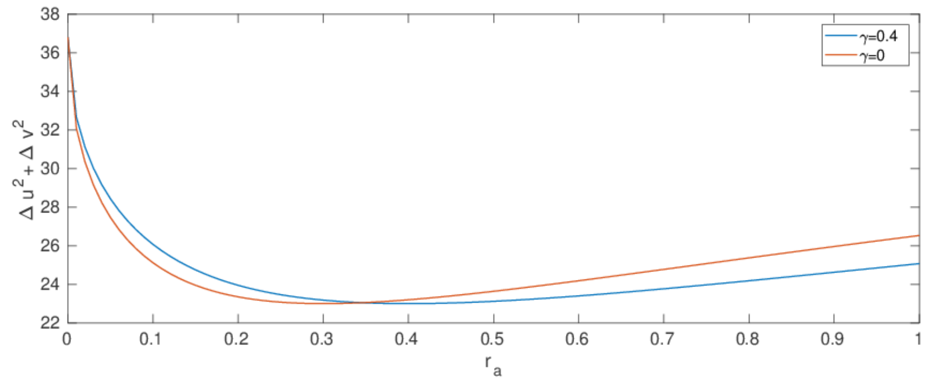
We note that, unlike the mean photon number, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the cavity light is independent of the number of photons. The plots in Fig. 5 and 6 indicate that the quadrature squeezing is greater for $\gamma = 0$ than that for $\gamma = 0.4$ for $0.01 < r_a < 0.35$ and is smaller for $\gamma = 0$ than that for $\gamma = 0.4$ for $0.35 < r_a < 1$. In addition, from the plots we see that the maximum quadrature squeezing is 37.5% both for $\gamma = 0$ and $\gamma = 0.4$. This occurs when the three-level laser is operating at $r_a = 0.30$ and $r_a = 0.40$, respectively. This result is less than the one obtained by Menisha[17] .

## 5. Entanglement Properties of the Two-Mode Light

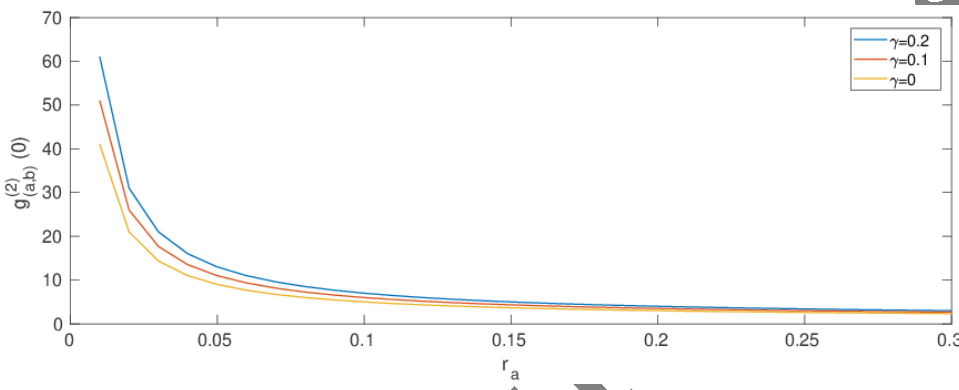
Here we proceed to study the entanglement condition of the two modes in the cavity. A pair of particles is taken to be entangled in quantum theory, if its states cannot be expressed as a product of the states of its individual constituents. The preparation and manipulation of these entangled states that have nonclassical and nonlocal properties lead to a better understanding of the basic quantum principles [16-20]. That is, if the density operator for the combined state cannot be described as a combination of the product of density operators of the constituents,

$$\hat{\rho} \neq \sum_j P_j \hat{\rho}_j^1 \otimes \hat{\rho}_j^2, \tag{166}$$

where $P_j \geq 0$ and $\sum_j P_j = 1$ is set to ensure normalization of the combined density of state. Nowadays, a lot of criteria have been developed to measure, detect, and manipulate the entanglement generated by various quantum optical devices. According to DGCZ[18] , a quantum state of a system is said to be entangled if the sum of the variances of the EPR-like quadrature operators, $\hat{u}$ and $\hat{v}$, satisfy the inequality




Φιγυρε 7: Plots of $\Delta u^2+\Delta v^2$ of the two-mode cavity light versus $r_a$ for $\kappa=0.2$, $\gamma_c=1.2$, and different values of $\gamma$.




Φιγυρε 8: Plots of $g^{(2)}_{(a,b)}(0)$ of the two-mode cavity light versus $r_a$ for $\gamma_c=0.4$ and for different values of $\gamma$.

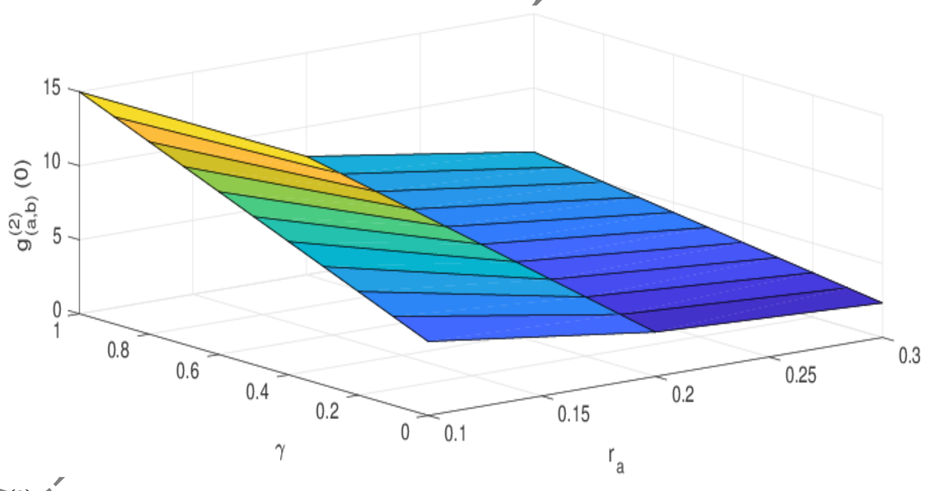



Figure 9: Plots of $g^{(2)}_{(a,b)}(0)$ of the two-mode cavity light versus $r_a$ and $\gamma$ for $\gamma_c=0.4$.

$(\Delta\hat{u})^2+(\Delta\hat{v})^2<2N,$ (167)
where
$\hat{u}=\hat{x}_a-\hat{x}_b,$ (168)
$\hat{v}=\hat{p}_a+\hat{p}_b,$ (169)
where $\hat{x}_a=(\hat{a}^\dagger+\hat{a})/\sqrt{2}$, $\hat{x}_b=(\hat{b}^\dagger+\hat{b})/\sqrt{2}$ $\hat{p}_a=i(\hat{a}^\dagger-\hat{a})/\sqrt{2}$, $\hat{p}_b=i(\hat{b}^\dagger-\hat{b})/\sqrt{2}$, are quadrature operators for modes $\hat{a}$ and $\hat{b}$. Taking into account [168] and [169], [167] yields
$(\Delta\hat{u})^2+(\Delta\hat{v})^2=2\frac{\gamma_c}{\kappa}[N+\langle\hat{N}_b\rangle-\langle\hat{m}_c\rangle].$ (170)
Thus, in view of equation [170] together with [160] and [161], the sum of the variances of $\hat{u}$ and $\hat{v}$ can be expressed as

$(\Delta\hat{u})^2+(\Delta\hat{v})^2=2\Delta c_-^2,$ (171)
where $\Delta c_-^2$ given by [161]. One can readily see from this result that the degree of entanglement is directly proportional to the degree of squeezing of the two- mode light. One can immediately notice that this particular entanglement measure is directly related the two-mode squeezing. This direct relationship shows that, whenever there is a two-mode squeezing in the system, there will be entanglement in the system as well. It is worth to note that the entanglement disappears when the squeezing vanishes. This is due to the fact that the entanglement is directly related to the squeezing, as given by [161]. It also follows that, like the mean photon number and quadrature

variance, the degree of entanglement depends on the number of atoms. With the help of criterion [168], we get that a significant entanglement occurs between the states of the light generated in the cavity. This is due to the strong correlation between the radiation emitted, when the atoms decay from the upper energy level to the lower via the intermediate level. In figure 7, the sum of the variances of a pair of EPR-type operators $\Delta\hat{u}^2 + \Delta\hat{v}^2$ is plotted against the pumping rate so that the available entanglement is clearly evident for various values of the spontaneous emission rate, $\gamma$.

## 6. Normalized Second-Order Correlation Functions

The second-order correlation function for the superposition of the two modes of the cavity radiation at equal time, can also be investigated, by using [18-21]:

$$g^{(2)}_{(a,b)}(0) = \frac{\langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle}. \tag{172}$$

Since $\hat{a}$ and $\hat{b}$ are Gaussian variables with vanishing means, the normalized second-order correlation function for the two-mode light takes, at the steady-state, the form

$$g^{(2)}_{(a,b)}(0) = 1 + \frac{\langle \hat{b}\hat{a} \rangle \langle \hat{a}^\dagger \hat{b}^\dagger \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle}. \tag{173}$$

It then follows that

$$g^{(2)}_{(a,b)}(0) = 1 + \frac{\langle \hat{m}_c \rangle^2}{\langle \bar{N}_a \rangle \langle \bar{N}_b \rangle}. \tag{174}$$

In view of [82], [83], and [147], we obtain

$$g^{(2)}_{(a,b)}(0) = 1 + \frac{\gamma_c + \gamma}{r_a}. \tag{175}$$

It can be seen from this result that the second-order correlation function of the two-mode light does not depend on the number of atoms.

Figure 8 and 9 shows that the second-order correlation function for the two-mode light versus $r_a$ in the presence $(\gamma \neq 0)$ and absence $(\gamma = 0)$ of the spontaneous emission. One can see from this figure that $g^{(2)}_{a,b}(0)$ decreases, as $r_a$ increases in both cases. It can be observed from the same figure that the second-order correlation function vanishes for $r_a < 0.01$. Moreover, the effect of the spontaneous emission increases the second-order correlation function.

## 7. Conclusion

In this paper we have studied the squeezing and entanglement properties of the light generated by three-level atoms available in an open cavity and pumped to the top level by electron bombardment at constant rate. Applying the large-time approximation scheme, we have obtained the steady-state solutions of the equations of evolution for the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators.

Using the resulting steady-state solutions, we have calculated the mean photon number, the variance of the photon number, the quadrature variance, quadrature squeezing, and entanglement for the two-mode cavity light. In addition, the normalized second-order correlation function is obtained for the superposition of the two modes. We have seen that the light generated by the three-level laser is in a squeezed state and the squeezing occurs in the minus quadrature. It so turns out that the maximum quadrature squeezing of the two-mode cavity light is 37.5% for $\gamma = 0$ and $\gamma = 0.4$ below the vacuum-state level. Our result shows that the maximum quadrature squeezing is less than the one obtained by[10,17] . This is due to the vacuum reservoir noise. In addition, we have shown that the intracavity quadrature squeezing is enhanced due to the spontaneous emission. It is found that the squeezing and entanglement in the two-mode light are directly related. As a result, an increase in the degree of squeezing directly leads to an increase in the degree of entanglement and vice versa. This shows that, whenever there is squeezing in the two-mode light, there exists an entanglement in the system.

## 5. References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