IRANIAN JOURNAL OF PHYSICS RESEARCH

Iranian Journal of Physics Research, Vol. 24, No. 3, 2024
DOI: 10.47176/ijpr.24.3.21416

Quantum properties of nondegenerate three-level laser coupled vacuum reservoir

M Alemu

Department of Physics, Wolkite University P. O. Box 07, Wolkite, Ethiopia

E-mail: minisha.alemu @ gmail.com

(Received 14 February 2022 ; in final form 23 August 2023)

Abstract

In this paper, we have studied the squeezing and statistical properties of the light produced by a three-level laser whose cavity contains a parametric amplifier, and with the cavity mode driven by coherent light and coupled to a vacuum reservoir. We obtain stochastic differential equations associated with the normal ordering using the pertinent master equation. By making use of the solutions to the resulting differential equations, we calculate the quadrature variances. We also determine the mean and variance of the photon number for the cavity mode by employing the Q function. It is found that the parametric amplifier increases the degree of squeezing, while the driving coherent light does not have any effect on the squeezing. Moreover, the mean photon number increases considerably due to the driving coherent light and the parametric amplifier.

Keywords: stochastic differential equations, C-number Langevin equations, vacuum reservoir, mean photon number

1. Introduction

There has been a considerable interest in the analysis of the squeezing and statistical properties of the light generated by three-level lasers [1-20]. A light mode to be in a squeezed state, if either the change in plus quadrature or the change in minus quadrature is less than one, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation. Because of a less noise in one quadrature component, the squeezed states of light have important applications in information processing systems like quantum computations, photon detection, and in the field of high-precision measurements [10 - 20].
A three-level laser may be defined as a quantum optical system in which the injected three-level atoms in a cascade configuration are initially prepared in a coherent superposition of the top and bottom levels and coupled to a vacuum reservoir via a single port mirror. When a three-level atom in a cascade configuration makes a transition from the top to the bottom level via the intermediate level, two photons are generated. If the two photons have the same frequency, then the three-level atom is called degenerate three-level atom; o\herwise it is called nondegenerate [3]. The two photons are highly correlated and this correlation is responsible for the production of squeezed light.
Three-level lasers in which the crucial role is played by the coherent superposition of the top and bottom levels of the injected atoms have been studied by several authors [1-7]. These studies show that this quantum optical system can generate light in a squeezed state under certain conditions. Currently, Menisha [21] has studied the squeezing and statistical properties of the cavity modes produced by two nondegenerate three-level atoms, with the cavity mode coupled to a vacuum reservoir. He has shown that the maximum quadrature squeezing of the light generated by the laser for A = 100 and κ = 0.8, is found to be 65.3% below the coherent-state level.
In addition, Fesseha has studied the squeezing and statistical properties of the light produced by a degenerate three-level laser whose cavity contains a degenerate parametric amplifier [4]. His study indicates that a more squeezed light could be generated by a combination of these two quantum optical systems. On the other hand, Alebachew and Fesseha [10] have considered the same system with the injected atoms having equal probability to be in the upper and lower levels and with these two levels coupled by the pump mode emerging from the parametric amplifier. This study shows that the system generates light in a squeezed state with a maximum intracavity squeezing of 93% below the coherentstate level.
In this paper, we introduce a model that generates bright and squeezed light from a two nondegenerate three-level atoms, in which the cavity modes contains a parametric amplifier and with the cavity mode driven by coherent light and coupled to a vacuum reservoir. We consider a two nondegenerate three-level laser in which the pump mode emerging from the parametric amplifier does not couple the top and bottom levels of the injected atoms.

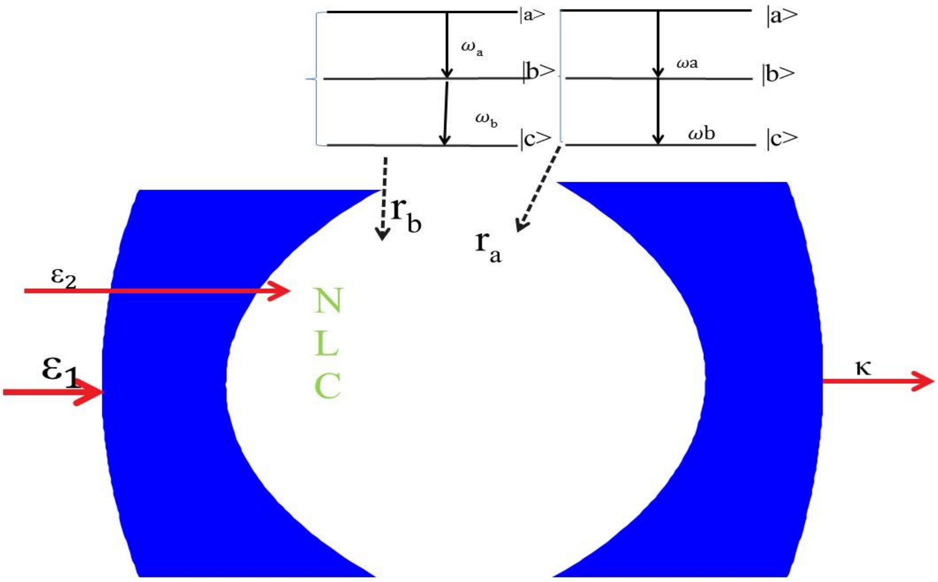

Figure 1. Schematic representation of two nondegenerate three-level lasers.

The two atoms are different in preparation and injection rate (see figure 1). In order to determine the squeezing and statistical properties of the light produced by this quantum optical system, we first derive c-number Langevin equations using the pertinent master equation. Employing the solutions of the resulting c-number Langevin equations along with the properties of the Langevin forces, we calculate the quadrature variance of the cavity mode. Applying the same solutions, we also obtain the antinormally ordered characteristic function with the aid of which the Q function is determined. The resulting Q function is then used to calculate the mean and variance of the photon number sum and difference of the cavity mode.

2. Stochastic differential equations

In this section we consider a two nondegenerate three-level laser whose cavity contains a nondegenerate parametric amplifier (NOPA) and with the cavity modes driven by a strong coherent light and coupled to a vacuum reservoir. The three-level atoms injected into the cavity are initially prepared in a coherent superposition of the top and bottom levels. As it is clearly indicated in Figure 1, the top, intermediate, and bottom levels of a three-level atom are represented by |*a*>, |*b*>, and |*c*>, respectively. We prefer to call the light emitted from the top level light mode *a* and the one emitted from the intermediate level light mode *b*. We assume the transitions between levels |*a*> and |*b*> and between levels |*b*> and |*c*> to be dipole allowed, with direct transitions between levels |*a*i and level |*c*i to be dipole forbidden. We consider the case for which the two cavity modes are at resonance with the two transitions |*a*> → |*b*> and |*b*> → |*c*> having transition frequencies ω_a and ω_b, respectively. The interaction of a nondegenerate three-level atom with two-mode cavity radiation can be expressed in the interaction picture with the rotating-wave approximation (RWA) by the Hamiltonian of the form [3]

$$\hat{H} = ig\left(|a\rangle\langle b|\hat{a} - \hat{a}^{\dagger}|b\rangle\langle a| + |b\rangle\langle c|\hat{b} - \hat{b}^{\dagger}|c\rangle\langle b|\right) , \quad (1)$$

where g is the coupling constant assumed to be the same for both transitions, $\hat{a}$ and $\hat{b}$ are the annihilation operators for the cavity modes.

The interaction of the driving light modes, treated classically, and cavity modes is described by the Hamiltonian [4]

$$\hat{H} = i\varepsilon_1\left(\hat{a}^{\dagger} - \hat{a} + \hat{b}^{\dagger} - \hat{b}\right) , \quad (2)$$

where ε_1 is proportional to the amplitude of the driving light modes. In addition, the Hamiltonian describing the parametric interaction, with the pump mode treated classically, can be written as [10]

$$\hat{H} = i\varepsilon_2\left(\hat{a}^{\dagger}\hat{b}^{\dagger} - \hat{a}\hat{b}\right) , \quad (3)$$

where ε_2 is proportional to the amplitude of the pump mode. In this paper, we suppose the state of a single three-level atom initially in the state

$$|\psi_A(0)\rangle = C_a|a\rangle + C_c|c\rangle, \quad (4)$$

and hence, the density operator of a single atom is

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c| , \quad (5)$$

where

$$\rho_{aa}^{(0)} = |c_a|^2 \quad and \quad \rho_{cc}^{(0)} = |c_c|^2 , \quad (6)$$

are the initial probabilities of the atoms to be in the upper and lower levels, respectively, and

$$\rho_{ac}^{(0)} = C_a C_c^* \quad and \quad \rho_{ca}^{(0)} = C_c C_a^*, \quad (7)$$

represent the atomic coherence at the initial time. We note that

$$\left|\rho_{ac}^{(0)}\right|^2 = \rho_{aa}^{(0)}\,\rho_{cc}^{(0)} , \quad (8)$$

Following the straightforward procedure outlined in [4], the master equation for the cavity modes of a nondegenerate three-level laser whose cavity contains a nondegenerate parametric amplifier and whose cavity modes are driven by a two-mode coherent light and coupled to a two-mode vacuum reservoir can be written as

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= \varepsilon_1\left(\hat{\rho}\hat{a}-\hat{a}\hat{\rho}+\hat{a}^{\dagger}\hat{\rho}-\hat{\rho}\hat{a}^{\dagger}+\hat{\rho}\hat{b}-\hat{b}\hat{\rho}+\hat{b}^{\dagger}\hat{\rho}-\hat{\rho}\hat{b}^{\dagger}\right)\\ &+\varepsilon_2\left(\hat{\rho}\hat{a}\hat{b}-\hat{a}\hat{b}\hat{\rho}+\hat{a}^{\dagger}b^{\dagger}\hat{\rho}-\hat{\rho}\hat{a}^{\dagger}\hat{b}^{\dagger}\right)\\ &+\frac{1}{2}\left[\begin{array}{l}A_1\rho_{aa}^{(0)}\left(2\hat{a}^{\dagger}\hat{\rho}\hat{a}-\hat{a}\hat{a}^{\dagger}\hat{\rho}-\hat{\rho}\hat{a}\hat{a}^{\dagger}\right)\\ +\left(A_1\rho_{cc}^{(0)}+\kappa\right)\left(2\hat{b}\hat{\rho}\hat{b}^{\dagger}-\hat{\rho}\hat{b}^{\dagger}\hat{b}-\hat{b}^{\dagger}\hat{b}\hat{\rho}\right)\end{array}\right]\\ &-\frac{1}{2}\left[\begin{array}{l}A_1\rho_{ac}^{(0)}\left(2\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger}-\hat{b}^{\dagger}\hat{a}^{\dagger}\hat{\rho}-\hat{\rho}\hat{b}^{\dagger}\hat{a}^{\dagger}\right)\\ +A_1\rho_{ca}^{(0)}\left(2\hat{b}\hat{\rho}\hat{a}-\hat{\rho}\hat{a}\hat{b}-\hat{a}\hat{b}\hat{\rho}\right)\end{array}\right]\\ &+\frac{\kappa_1}{2}\left[\left(2\hat{a}\hat{\rho}\hat{a}^{\dagger}-\hat{a}^{\dagger}\hat{a}\hat{\rho}-\hat{\rho}\hat{a}^{\dagger}\hat{a}\right)\right],\end{aligned} \tag{9}$$

where

$$A_1=\frac{2g^2r_a}{\gamma_1^2}, \tag{10}$$

is the linear gain coefficient, κ_1 is a cavity damping constant, and γ_1 which is considered to be the same for all the three-levels, is the atomic decay constant. In view of eq. (9), we can also write the master equation for the cavity mode in which two different types of atoms injected at rates r_a and r_b as

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= \chi_1\left(\begin{array}{l}\hat{\rho}\hat{a}-\hat{a}\hat{\rho}+\hat{a}^{\dagger}\hat{\rho}-\hat{\rho}\hat{a}^{\dagger}+\hat{\rho}\hat{b}-\hat{b}\hat{\rho}+\\ \hat{b}^{\dagger}\hat{\rho}-\hat{\rho}\hat{b}^{\dagger}\end{array}\right)\\ &+\chi_2\left(\hat{\rho}\hat{a}\hat{b}-\hat{a}\hat{b}\hat{\rho}+\hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\rho}-\hat{\rho}\hat{a}^{\dagger}\hat{b}^{\dagger}\right)\\ &+\frac{A}{2}\{\rho_{aa}^{(0)}\left(2\hat{a}^{\dagger}\hat{\rho}\hat{a}-\hat{a}\hat{a}^{\dagger}\hat{\rho}-\hat{\rho}\hat{a}\hat{a}^{\dagger}\right)\\ &+\rho_{cc}^{(0)}\left(2\hat{b}\hat{\rho}\hat{b}^{\dagger}-\hat{\rho}\hat{b}^{\dagger}\hat{b}-\hat{b}^{\dagger}\hat{b}\hat{\rho}\right)\\ &-\rho_{ac}^{(0)}\left(2\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger}-\hat{b}^{\dagger}\hat{a}^{\dagger}\hat{\rho}-\hat{\rho}\hat{b}^{\dagger}\hat{a}^{\dagger}\right)\\ &-\rho_{ca}^{(0)}\left(2\hat{b}\hat{\rho}\hat{a}-\hat{\rho}\hat{a}\hat{b}-\hat{a}\hat{b}\hat{\rho}\right)\}\\ &+\frac{\kappa}{2}\left(\begin{array}{l}2\hat{a}\hat{\rho}\hat{a}^{\dagger}-\hat{a}^{\dagger}\hat{a}\hat{\rho}-\hat{\rho}\hat{a}^{\dagger}\hat{a}+2\hat{b}\hat{\rho}\hat{b}^{\dagger}-\\ \hat{b}^{\dagger}\hat{b}\hat{\rho}-\hat{\rho}\hat{b}^{\dagger}\hat{b}\end{array}\right),\end{aligned} \tag{11}$$

where

$$\chi_1=\varepsilon_1+\varepsilon_1', \tag{12}$$

$$\chi_2=\varepsilon_2+\varepsilon_2', \tag{13}$$

$$A=A_1+A_1', \tag{14}$$

and

$$\kappa=\kappa_1+\kappa_1', \tag{15}$$

3. C-number Langevin equations

We now seek to obtain the c-number Langevin equations associated with the normal ordering for the cavity mode variables. To this end, employing the relation [5]

$$\frac{d}{dt}\langle\hat{B}\rangle=Tr\left(\frac{d\hat{\rho}(t)}{dt}\hat{B}\right), \tag{16}$$

along with eq. (11), one readily finds the following equations

$$\frac{d}{dt}\langle\hat{a}\rangle=-\frac{1}{2}\mu_a\langle\hat{a}\rangle+\frac{1}{2}\nu_-\langle\hat{b}^{\dagger}\rangle+\chi_1, \tag{17}$$

$$\frac{d}{dt}\langle\hat{b}\rangle=-\frac{1}{2}\mu_c\langle\hat{b}\rangle+\frac{1}{2}\nu_+\langle\hat{a}^{\dagger}\rangle+\chi_1, \tag{18}$$

$$\frac{d}{dt}\langle\hat{a}^2\rangle=-\mu_a\langle\hat{a}^2\rangle+\nu_-\langle\hat{b}^{\dagger}\hat{a}\rangle+2\chi_1\langle\hat{a}\rangle, \tag{19}$$

$$\frac{d}{dt}\langle\hat{b}^2\rangle=-\mu_c\langle\hat{b}^2\rangle+\nu_+\langle\hat{a}^{\dagger}\hat{b}\rangle+2\chi_1\langle\hat{b}\rangle, \tag{20}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^{\dagger}\hat{a}\rangle&=-\mu_a\langle\hat{a}^{\dagger}\hat{a}\rangle+\frac{1}{2}\nu_-\langle\hat{a}^{\dagger}\hat{b}^{\dagger}\rangle+\\ &\frac{1}{2}\nu_-^*\langle\hat{a}\hat{b}\rangle+\chi_1\left(\langle\hat{a}\rangle+\langle\hat{a}^{\dagger}\rangle\right)+A\rho_{aa}^{(0)},\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{b}^{\dagger}\hat{b}\rangle&=-\mu_c\langle\hat{b}^{\dagger}\hat{b}\rangle+\frac{1}{2}\nu_+\langle\hat{b}^{\dagger}\hat{a}^{\dagger}\rangle\\ &+\frac{1}{2}\nu_+^*\langle\hat{a}\hat{b}\rangle+\chi_1\left(\langle\hat{b}\rangle+\langle\hat{b}^{\dagger}\rangle\right),\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^{\dagger}\hat{b}\rangle&=-\frac{1}{2}(\mu_a+\mu_c)\langle\hat{a}^{\dagger}\hat{b}\rangle+\frac{1}{2}\nu_+\langle\hat{a}^{\dagger 2}\rangle\\ &+\frac{1}{2}\nu_-^*\langle\hat{b}^2\rangle+\chi_1\left(\langle\hat{b}\rangle+\langle\hat{a}^{\dagger}\rangle\right),\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\hat{b}\rangle&=-\frac{1}{2}(\mu_a+\mu_c)\langle\hat{a}\hat{b}\rangle+\frac{1}{2}\nu_+\langle\hat{a}^{\dagger}\hat{a}\rangle\\ &+\frac{1}{2}\nu_-\langle\hat{b}^{\dagger}\hat{b}\rangle+\chi_1\left(\langle\hat{b}\rangle+\langle\hat{a}\rangle\right)+\frac{1}{2}\nu_+,\end{aligned} \tag{24}$$

where

$$\mu_a=\kappa-A\rho_{aa}^{(0)}, \tag{25}$$

$$\mu_c=\kappa+A\rho_{cc}^{(0)}, \tag{26}$$

$$\nu_-=2\chi_2-A\rho_{ac}^{(0)}, \tag{27}$$

$$\nu_+=2\chi_2+A\rho_{aa}^{(0)}, \tag{28}$$

We note that the operators in the above equations are in the normal order. The c-number equations corresponding to eqs. (17-24) are [2]

$$\frac{d}{dt}\langle\alpha\rangle=-\frac{1}{2}\mu_a\langle\alpha\rangle+\frac{1}{2}\nu_-\langle\beta^*\rangle+\chi_1, \tag{29}$$

$$\frac{d}{dt}\langle\beta\rangle=-\frac{1}{2}\mu_c\langle\beta\rangle+\frac{1}{2}\nu_+\langle\alpha^*\rangle+\chi_1, \tag{30}$$

$$\frac{d}{dt}\langle\alpha^2\rangle=-\mu_a\langle\alpha^2\rangle+\nu_-\langle\beta^*\alpha\rangle+2\chi_1\langle\alpha\rangle, \tag{31}$$

$$\frac{d}{dt}\langle\beta^2\rangle=-\mu_c\langle\beta^2\rangle+\nu_+\langle\alpha^*\beta\rangle+2\chi_1\langle\beta\rangle, \tag{32}$$

$$\begin{aligned}\frac{d}{dt}\langle\alpha^*\alpha\rangle&=-\mu_a\langle\alpha^*\alpha\rangle+\frac{1}{2}\nu_-\langle\alpha^*\beta^*\rangle\\ &+\frac{1}{2}\nu_-^*\langle\alpha\beta\rangle+\chi_1\left(\langle\alpha\rangle+\langle\alpha^*\rangle\right)+A\rho_{aa}^{(0)},\end{aligned} \tag{33}$$

$$\frac{d}{dt}\left\langle \beta^{*}\beta\right\rangle=-\mu_{c}\left\langle \beta^{*}\beta\right\rangle+\frac{1}{2}\nu_{+}\left\langle \beta^{*}\alpha^{*}\right\rangle +\frac{1}{2}\nu_{+}^{*}\langle\alpha\beta\rangle+\chi_{1}\left(\langle\beta\rangle+\left\langle\beta^{*}\right\rangle\right) , \tag{34}$$

$$\frac{d}{dt}\left\langle \alpha^{*}\beta\right\rangle=-\frac{1}{2}\langle\mu_{a}+\mu_{c}\rangle\left\langle \alpha^{*}\beta\right\rangle+\frac{1}{2}\nu +\left\langle \alpha^{*2}\right\rangle+\frac{1}{2}\nu_{-}^{*}\left\langle \beta^{2}\right\rangle+\chi_{1}\left(\langle\beta\rangle+\left\langle\alpha^{*}\right\rangle\right) , \tag{35}$$

$$\frac{d}{dt}\langle\alpha\beta\rangle=-\frac{1}{2}\langle\mu_{a}+\mu_{c}\rangle\langle\alpha\beta\rangle+\frac{1}{2}\nu_{+}\left\langle \alpha^{*}\alpha\right\rangle +\frac{1}{2}\nu_{-}\left\langle \beta^{*}\beta\right\rangle+\chi_{1}\left(\langle\beta\rangle+\langle\alpha\rangle\right)+\frac{1}{2}\nu_{+} , \tag{36}$$

On the basis of eqs. (29) and (30), we can write

$$\frac{d}{dt}\alpha(t)=-\frac{1}{2}\mu_{a}\alpha(t)+\frac{1}{2}\nu_{-}\beta^{*}(t)+\chi_{1}+f_{\alpha}(t) , \tag{37}$$

$$\frac{d}{dt}\beta^{*}(t)=-\frac{1}{2}\mu_{c}\beta^{*}(t)+\frac{1}{2}\nu_{+}^{*}\alpha(t)+\chi_{1}+f_{\beta}^{*}(t) , \tag{38}$$

where $f_\alpha(t)$ and $f_\beta(t)$ are Langevin forces the properties of which remain to be determined, $\alpha(t)$ and $\beta(t)$ are the c-number variables corresponding to the cavity mode operators $\hat{a}$ and $\hat{b}$. Making the use of eqs. (29, 30), the correlation properties of the Langevin forces can be readily put as [3]

$$\left\langle f_{\alpha}(t)\right\rangle=\left\langle f_{\beta}(t)\right\rangle=0 , \tag{39}$$

$$\left\langle f_{\alpha}(t')f_{\alpha}(t)\right\rangle=0 , \tag{40}$$

$$\left\langle f_{\beta}(t')f_{\beta}(t)\right\rangle=\left\langle f_{\alpha}^{*}(t')f_{\beta}(t)\right\rangle=0 , \tag{41}$$

$$\left\langle f_{\alpha}^{*}(t')f_{\alpha}(t)\right\rangle=A\rho_{aa}^{(0)}\delta(t-t') , \tag{42}$$

$$\left\langle f_{\beta}^{*}(t')f_{\beta}(t)\right\rangle=0 , \tag{43}$$

$$\left\langle f_{\alpha}(t')f_{\beta}(t)\right\rangle=\frac{1}{2}\nu_{+}\delta(t-t') , \tag{44}$$

The results described by eqs. (39- 44) represent the correlation properties of the Langevin forces $f_\alpha(t)$ and $f_\beta(t)$ associated with the normal ordering. Following the procedure described in [20], the solutions of the coupled differential eqs. (37 , 38) are given by

$$\alpha(t)=p_{1}(t)\alpha(0)+q_{1}(t)\beta^{*}(0)+F_{a}(t)+\varepsilon_{11}(t) , \tag{45}$$

$$\beta^{*}(t)=p_{2}(t)\beta^{*}(0)+q_{2}(t)\alpha(0)+F_{b}(t)+\varepsilon_{12}(t) , \tag{46}$$

where

$$p_{1}(t)=\frac{A_{+}}{2\lambda}e^{\frac{-1}{2}\lambda_{-}t}-\frac{A_{-}}{2\lambda}e^{\frac{-1}{2}\lambda_{+}t} , \tag{47}$$

$$p_{2}(t)=\frac{A_{+}}{2\lambda}e^{\frac{-1}{2}\lambda_{+}t}-\frac{A_{-}}{2\lambda}e^{\frac{-1}{2}\lambda_{-}t} , \tag{48}$$

$$q_{1}(t)=\frac{2\upsilon_{-}}{2\lambda}e^{\frac{-1}{2}\lambda_{-}t}-\frac{2\upsilon_{-}}{2\lambda}e^{\frac{-1}{2}\lambda_{+}t} , \tag{49}$$

$$q_{2}(t)=\frac{2\upsilon_{+}^{*}}{2\lambda}e^{\frac{-1}{2}\lambda_{-}t}-\frac{2\upsilon_{+}^{*}}{2\lambda}e^{\frac{-1}{2}\lambda_{+}t} , \tag{50}$$

$$F_{a}(t)=\int_{0}^{t}\begin{bmatrix}p_{1}(t-t')f_{\alpha}(t')+\\ q_{1}(t-t')f_{\beta^{*}}(t')\end{bmatrix}dt' , \tag{51}$$

$$F_{b}(t)=\int_{0}^{t}\begin{bmatrix}q_{1}(t-t')f_{\beta^{*}}(t')+\\ q_{2}(t-t')f_{\alpha}(t')\end{bmatrix}dt' , \tag{52}$$

$$\varepsilon_{11}(t)=\frac{\chi_{1}}{\lambda}\begin{bmatrix}\frac{A_{+}+2\upsilon_{-}}{\lambda_{-}}\left(1-e^{\frac{-1}{2}\lambda_{-}t}\right)\\ -\frac{A_{-}+2\upsilon_{-}}{\lambda_{+}}\left(1-e^{\frac{-1}{2}\lambda_{+}t}\right)\end{bmatrix} , \tag{53}$$

$$\varepsilon_{12}(t)=\frac{\chi_{1}}{\lambda}\begin{bmatrix}\frac{A_{+}-2\upsilon_{-}}{\lambda_{+}}\left(1-e^{\frac{-1}{2}\lambda_{+}t}\right)\\ -\frac{A_{-}-2\upsilon_{-}}{\lambda_{-}}\left(1-e^{\frac{-1}{2}\lambda_{-}t}\right)\end{bmatrix} , \tag{54}$$

$$\lambda_{\pm}=\frac{1}{2}\left[(\mu_{a}+\mu_{c})\pm\sqrt{(\mu_{a}-\mu_{c})^{2}+4\upsilon_{+}^{*}\upsilon_{-}}\right] , \tag{55}$$

$$\lambda=\sqrt{A^{2}+4\upsilon_{+}^{*}\upsilon_{-}} , \tag{56}$$

$$A_{\pm}=A\pm\lambda . \tag{57}$$

4. Quadrature variance of the cavity modes

Here, we seek to analyze the quadrature squeezing of the two-mode light in the cavity. The squeezing properties of the two-mode light in the cavity can be described by two quadrature operators defined by [18]

$$\hat{c}_{\pm}=\sqrt{\pm1}\left(\hat{c}^{\dagger}\pm\hat{c}\right) , \tag{58}$$

where

$$\hat{c}=\frac{1}{\sqrt{2}}\left(\hat{a}+\hat{b}\right), \tag{59}$$

with $\hat{a}$ and $\hat{b}$ represent the separate modes of cavity light emitted from the three-level atoms. The two-mode light is said to be in a squeezed state if either $\Delta c_{+}^{2}<1$ and $\Delta c_{-}^{2}>1$ or $\Delta c_{+}^{2}>1$ and $\Delta c_{1}^{2}<1$, such that $\Delta c_{+}\Delta c_{-}\geq 1$ [3, 20].

The variances of the quadrature operator, defined by

$$\Delta c_{\pm}^{2}=\langle\hat{c}_{\pm}^{2}\rangle-\langle\hat{c}_{\pm}\rangle^{2} , \tag{60}$$

can be expressed in terms of c-number variables associated with the normal ordering as

$$\Delta c_{\pm}^{2}=1\pm\langle\gamma_{\pm}(t),\gamma_{\pm}(t)\rangle , \tag{61}$$

where

$$\gamma_{\pm}(t)=\frac{1}{\sqrt{2}}\left(\alpha^{*}(t)+\beta^{*}(t)\pm\alpha(t)\pm\beta(t)\right) , \tag{62}$$

On account of Eq. (62), we see that

$$\langle \gamma_{\pm}(t),\gamma_{\pm}(t)\rangle = \frac{1}{2}(\langle \alpha(t),\alpha(t)\rangle + \langle \beta^*(t),\beta^*(t)\rangle + 2\langle \alpha(t),\beta(t)\rangle \pm \langle \alpha^*(t),\alpha(t)\rangle \pm \langle \beta^*(t),\beta(t)\rangle \pm 2\langle \beta^*(t),\alpha(t)\rangle) + c.c \quad , \tag{63}$$

in which *c.c.* stands for complex conjugate. Using eqs. (51, 52), (45, 46), and assuming the cavity modes are initially in vacuum states along with the fact that a noise force at a certain time does not affect the cavity mode variables at earlier time, one can easily establish that

$$\langle \alpha(t),\alpha(t)\rangle = \langle \beta(t),\beta(t)\rangle = \langle \beta^*(t),\alpha(t)\rangle = 0 \quad , \tag{64}$$

$$\begin{aligned}
\langle \alpha^*(t),\alpha(t)\rangle = {} & \frac{A_+^*[A_+ f_{\alpha\alpha} + 2\nu_- f^*_{\alpha\beta}] + 2\nu_-^* A_+ f_{\alpha\beta}}{|2\lambda|^2(\lambda_-^* + \lambda_-)/2}(1-e^{-\frac{1}{2}(\lambda_-^*+\lambda_-)t}) \\
& - \frac{A_+^*[A_- f_{\alpha\alpha} + 2\nu_- f^*_{\alpha\beta}] + 2\nu_-^* A_- f_{\alpha\beta}}{|2\lambda|^2(\lambda_-^* + \lambda_+)/2}(1-e^{-\frac{1}{2}(\lambda_-^*+\lambda_+)t}), \\
& - \frac{A_-^*[A_+ f_{\alpha\alpha} + 2\nu_- f^*_{\alpha\beta}] + 2\nu_-^* A_+ f_{\alpha\beta}}{|2\lambda|^2(\lambda_+^* + \lambda_-)/2}(1-e^{-\frac{1}{2}(\lambda_+^*+\lambda_-)t}) \\
& + \frac{A_-^*[A_- f_{\alpha\alpha} + 2\nu_- f^*_{\alpha\beta}] + 2\nu_-^* A_- f_{\alpha\beta}}{|2\lambda|^2(\lambda_+^* + \lambda_+)/2}(1-e^{-\frac{1}{2}(\lambda_+^*+\lambda_+)t}) \ ,
\end{aligned} \tag{65}$$

$$\begin{aligned}
\langle \beta^*(t),\beta(t)\rangle = {} & -\frac{A_+^* 2\nu_+^* f_{\alpha\beta} - 2\nu_+[A_+ f^*_{\alpha\beta} + 2\nu_-^* f_{\alpha^*\alpha}]}{|2\lambda|^2(\lambda_+^* + \lambda_+)/2}(1-e^{-\frac{1}{2}(\lambda_+^*+\lambda_+)t}) \\
& + \frac{A_+^* 2\nu_+^* f_{\alpha\beta} - 2\nu_+[A_- f^*_{\alpha\beta} + 2\nu_-^* f_{\alpha^*\alpha}]}{|2\lambda|^2(\lambda_+^* + \lambda_-)/2}(1-e^{-\frac{1}{2}(\lambda_+^*+\lambda_-)t}) \\
& + \frac{A_-^* 2\nu_+^* f_{\alpha\beta} - 2\nu_+[A_+ f^*_{\alpha\beta} + 2\nu_-^* f_{\alpha^*\alpha}]}{|2\lambda|^2(\lambda_-^* + \lambda_+)/2}(1-e^{-\frac{1}{2}(\lambda_-^*+\lambda_+)t}) \\
& - \frac{A_-^* 2\nu_+^* f_{\alpha\beta} - 2\nu_+[A_- f^*_{\alpha\beta} + 2\nu_-^* f_{\alpha^*\alpha}]}{|2\lambda|^2(\lambda_-^* + \lambda_-)/2}(1-e^{-\frac{1}{2}(\lambda_-^*+\lambda_-)t}) \ ,
\end{aligned} \tag{66}$$

and

$$\begin{aligned}
\langle \alpha(t),\beta(t)\rangle = {} & \frac{A_+^* A_+ f_{\alpha\beta} - 2\nu_+[A_+ f_{\alpha^*\alpha} + 2\nu_- f^*_{\alpha\beta}]}{|2\lambda|^2(\lambda_+^* + \lambda_-)/2}(1-e^{-\frac{1}{2}(\lambda_+^*+\lambda_-)t}) \\
& - \frac{A_+^* A_- f_{\alpha\beta} - 2\nu_+[A_- f_{\alpha^*\alpha} + 2\nu_- f^*_{\alpha\beta}]}{|2\lambda|^2(\lambda_+^* + \lambda_+)/2}(1-e^{-\frac{1}{2}(\lambda_+^*+\lambda_+)t}) \\
& - \frac{A_-^* A_+ f_{\alpha\beta} - 2\nu_+[A_+ f_{\alpha^*\alpha} + 2\nu_- f^*_{\alpha\beta}]}{|2\lambda|^2(\lambda_-^* + \lambda_-)/2}(1-e^{-\frac{1}{2}(\lambda_-^*+\lambda_-)t}) \\
& + \frac{A_-^* A_- f_{\alpha\beta} - 2\nu_+[A_- f_{\alpha^*\alpha} + 2\nu_- f^*_{\alpha\beta}]}{|2\lambda|^2(\lambda_-^* + \lambda_+)/2}(1-e^{-\frac{1}{2}(\lambda_-^*+\lambda_+)t}) \ ,
\end{aligned} \tag{67}$$

where

$$f_{\alpha\beta} = \frac{\upsilon_+}{2}, \qquad f_{\alpha^*\alpha} = A\rho^{(0)}_{aa} \ , \tag{68}$$

Now substitution of eqs. (64), (65), (66),and (67), and the complex conjugate of (67) into eq. (63) leads to

$$\begin{aligned}
\langle \gamma_{\pm}(t),\gamma_{\pm}(t)\rangle = {} & \pm\frac{1}{|2\lambda|^2}\Big[\frac{(A_+ \pm 2\nu_+^*)[(A_+^* \pm 2\nu_+)f_{\alpha^*\alpha} \mp (A_-^* \mp 2\nu_-^*)f_{\alpha\beta}]}{\lambda_- + \lambda_-^*} \mp \frac{(A_- \mp 2\nu_-)(A_+^* \pm 2\nu_+)f^*_{\alpha\beta}}{\lambda_- + \lambda_-^*}\Big](1-e^{-\frac{1}{2}(\lambda_-+\lambda_-^*)t}) \\
& \pm\frac{1}{|2\lambda|^2}\Big[\frac{(A_- \pm 2\nu_+^*)[(A_+^* \pm 2\nu_+)f_{\alpha^*\alpha} \mp (A_+^* \mp 2\nu_-^*)f_{\alpha\beta}]}{\lambda_+ + \lambda_+^*} \mp \frac{(A_- \mp 2\nu_-)(A_-^* \pm 2\nu_+)f^*_{\alpha\beta}}{\lambda_+ + \lambda_+^*}\Big](1-e^{-\frac{1}{2}(\lambda_++\lambda_+^*)t}) \\
& \mp\frac{2}{|2\lambda|^2}\Big[\frac{(A_+ \pm 2\nu_+^*)[(A_+^* \pm 2\nu_+)f_{\alpha^*\alpha} \mp (A_-^* \mp 2\nu_-^*)f_{\alpha\beta}]}{\lambda_+ + \lambda_-^*} \mp \frac{(A_- \mp 2\nu_-)(A_+^* \pm 2\nu_+)f^*_{\alpha\beta}}{\lambda_+ + \lambda_-^*}\Big](1-e^{-\frac{1}{2}(\lambda_++\lambda_-^*)t}) + c.c \ .
\end{aligned} \tag{69}$$

Hence on account of eq. (69), eq. (61) takes at steady state the form

$$\Delta c_{\pm}^{2}=1+\frac{2}{|2\lambda|^{2}}\Big[\frac{\left|A_{+}\pm 2\nu_{+}^{*}\right|^{2}}{\lambda_{-}+\lambda_{-}^{*}}+\frac{\left|A_{-}\pm 2\nu_{+}^{*}\right|^{2}}{\lambda_{+}+\lambda_{+}^{*}}-\frac{(A_{+}^{*}\pm 2\nu_{+})(A_{-}\pm 2\nu_{+}^{*})}{\lambda_{+}+\lambda_{-}^{*}}-\frac{(A_{+}\pm 2\nu_{+}^{*})(A_{-}^{*}\pm 2\nu_{+})}{\lambda_{+}^{*}+\lambda_{-}}\Big]f_{\alpha^{*}\alpha}$$

$$\mp\frac{2}{|2\lambda|^{2}}\Big[\frac{(A_{+}\pm 2\nu_{+}^{*})(A_{-}^{*}\mp 2\nu_{-}^{*})}{\lambda_{-}+\lambda_{-}^{*}}+\frac{(A_{+}^{*}\mp 2\nu_{-}^{*})(A_{-}\pm 2\nu_{+}^{*})}{\lambda_{+}+\lambda_{+}^{*}}-\frac{(A_{-}^{*}\mp 2\nu_{-}^{*})(A_{-}\pm 2\nu_{+}^{*})}{\lambda_{+}+\lambda_{-}^{*}}-\frac{(A_{+}^{*}\mp 2\nu_{-}^{*})(A_{+}\pm 2\nu_{+}^{*})}{\lambda_{+}^{*}+\lambda_{-}}\Big]f_{\alpha\beta}$$

$$\mp\frac{2}{|2\lambda|^{2}}\Big[\frac{(A_{+}^{*}\pm 2\nu_{+})(A_{-}\mp 2\nu_{-})}{\lambda_{-}+\lambda_{-}^{*}}+\frac{(A_{+}\mp 2\nu_{-})(A_{-}^{*}\pm 2\nu_{+})}{\lambda_{+}+\lambda_{+}^{*}}-\frac{(A_{+}\mp 2\nu_{-})(A_{+}^{*}\pm 2\nu_{+})}{\lambda_{+}+\lambda_{-}^{*}}-\frac{(A_{-}^{*}\pm 2\nu_{+})(A_{-}\mp 2\nu_{-})}{\lambda_{+}^{*}+\lambda_{-}}\Big]f_{\alpha^{*}\alpha}\,. \quad (70)$$

This represents the quadrature variances of the cavity modes for a two nondegenerate three- level laser whose cavity contains a parametric amplifier and whose cavity modes are driven by coherent light and coupled to a two-mode vacuum reservoir. In order to have a mathematically manageable analysis, we take $\rho_{ac}=\rho_{ca}$.
Now in view of this and eq. (8), we have

$$2\nu_{\pm}=2\nu_{\pm}^{*}=4\chi_{2}\pm A\sqrt{1-\eta^{2}}\ ,$$
$$\lambda=\lambda^{*}=\sqrt{A^{2}\eta^{2}+16\chi_{2}}\ ,$$
$$A_{\pm}=A_{\pm}^{*}=A\pm\sqrt{A^{2}\eta^{2}+16\chi_{2}}\ ,$$
$$\lambda_{\pm}=\lambda_{\pm}^{*}=\frac{1}{2}\big(2\kappa+A\eta\pm\sqrt{A^{2}\eta^{2}+16\chi_{2}}\,\big)\ , \quad (71)$$
$$f_{\alpha\beta}=f_{\alpha\beta}^{*}=\frac{(4\chi_{2}+A\sqrt{1-\eta^{2}}\,)}{4}\ ,$$
$$f_{\alpha^{*}\alpha}=\frac{A(1-\eta)}{2}\ .$$

So that with the aid eqs. (70, 71), we get

$$\Delta c_{\pm}^{2}=1+\frac{2\kappa A(1-\eta)(2\kappa+2A\eta+A)+16\chi_{2}^{2}A\eta}{4[\kappa(\kappa+A\eta)-4\chi_{2}^{2}](2\kappa+A\eta)}\pm\frac{2\kappa(4\chi_{2}+A\sqrt{1-\eta^{2}}\,)(2\kappa+A\eta+A\pm 4\chi_{2})}{4[\kappa(\kappa+A\eta)-4\chi_{2}^{2}](2\kappa+A\eta)}, \quad (72)$$

This is the quadrature variances of the cavity modes for a two nondegenerate three-level laser whose cavity contains a parametric amplifier and whose cavity modes are coupled to a two-mode vacuum reservoir. Since the parameter χ_{1} does not appear in this equation, the driving coherent light has no effect on the quadrature variances.

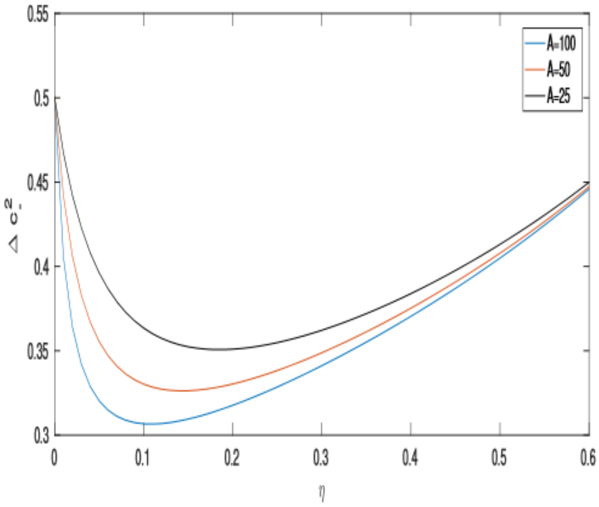

Figure 2. Plots of the quadrature variance [eq.(72)] versus η for $\kappa=0.8$, $\chi_{2}=0.399$, and for different values of the linear gain coefficient.

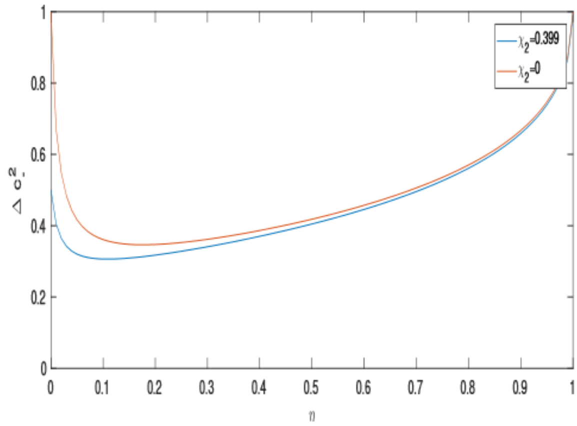

Figure 3. Plots of the quadrature variances [eq.(72)] versus η (blue curve) and [eq.(73)] versus η (red curve) for $A=100$, $\chi_{2}=0.399$, and $\kappa=0.8$.

Figure 2 represents the variances of the minus quadrature [eq. (72)] versus η for different values of the linear gain coefficient. This figure indicates that the degree of squeezing increases with the linear gain coefficient and almost perfect squeezing can be obtained for large values of the linear gain coefficient and for small values of η. Moreover, the minimum value of the quadrature variance described by eq. (72) for $A=100$, $\kappa=0.8$, and $X_{2}=0.399$, is found to be $\Delta c_{-}^{2}=0.3066$ and occurs at $\eta=0.11$. This result implies that the maximum intracavity squeezing for the above values is 69.34% below the coherent-state level. This result is greater than the one obtained by Menisha [21].
The plots in Figure 3 represent the variances of the minus quadrature of the cavity modes for a two nondegenerate three-level laser alone (red curve) and with parametric amplifier (blue curve). This figure indicates that better squeezing can be obtained from a two nondegenerate three-level laser with parametric amplifier. We now consider the case in which the nonlinear crystal is removed from the cavity and the cavity is coupled to a two-mode vacuum reservoir. Then upon setting $\chi_{2}=0$ in eq.(72), we get

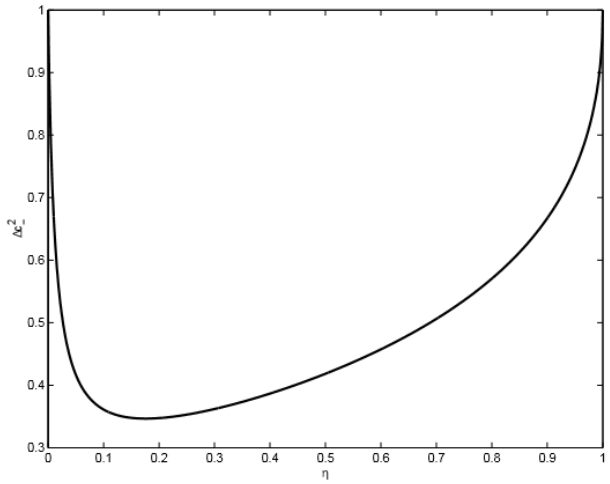

Figure 4. Plots of the quadrature variance [eq.(73)] versus η for A = 100 and κ = 0.8.

$$\Delta c_{\pm}^{2} = 1 \pm \frac{A\sqrt{1-\eta^{2}}\,(2\kappa + A\eta + A) \pm A(1-\eta)(2\kappa + 2A\eta + A)}{2(\kappa + A\eta)(2\kappa + A\eta)} \quad , \tag{73}$$

This is the quadrature variances of the cavity modes for a two nondegenerate three-level laser coupled to a two-mode vacuum reservoir. The minimum value of the quadrature variance described by figure 4 for A = 100 and κ = 0.8, is found to be $(\Delta c)_1^2 = 3467$ and occurs at η = 0.18. This result implies that the maximum intracavity squeezing for the above values is 65.3% below the coherent-state level. Comparison of this result with the 69.34% squeezing that could be obtained in the presence of the parametric amplifier shows that the parametric amplifier has significant effect on the squeezing of the cavity modes.

5. Photon statistics

In this section we study the statistical properties of the cavity modes produced by a two nondegenerate three-level laser whose cavity contains a parametric amplifier and with the cavity modes driven by coherent light and coupled to a two-mode vacuum reservoir. We first obtain, using the antinormally ordered characteristic function defined in the Heisenberg picture, the Q function for the cavity modes.

Then applying the resulting Q function, we calculate the mean and variances of the photon number sum and difference for the cavity modes.

5.1. The Q function

Here we wish to obtain the Q function for the cavity modes produced by the system under consideration. The Q function for a two-mode light can be expressed as [3]

$$Q(\alpha,\beta,t) = \frac{1}{\pi^{2}} \int \frac{d^{2}z}{\pi} \frac{d^{2}\omega}{\pi} \Phi_A(z,\omega,t)\; e^{z^{*}\alpha - z\alpha^{*} + \omega^{*}\beta - \omega\beta^{*}} \quad , \tag{74}$$

with the characteristic function $\Phi_A(z,w,t)$ defined in the Heisenberg picture by

$$\Phi_A(z,\omega,t) = Tr\;\big(\rho(0)\; e^{-z^{*}\hat{a}(t)}\; e^{z\hat{a}^{\dagger}(t)}\; e^{-\omega^{*}\hat{b}(t)}\; e^{\omega\hat{b}^{\dagger}(t)}\big) \;, \tag{75}$$

Employing the Baker-Hausdorff identity, we can rewrite eq. (75) in the normal order as

$$\Phi_A(z,\omega,t) = e^{-z^{*}z - \omega^{*}\omega}\; Tr\;\big(\rho(0)\; e^{z\hat{a}^{\dagger}(t)}\; e^{-z^{*}\hat{a}(t)}\; e^{\omega\hat{b}^{\dagger}(t)}\; e^{-\omega^{*}\hat{b}(t)}\big) \;, \tag{76}$$

so that the corresponding c-number equation is

$$\Phi_A(z,\omega,t) = e^{-z^{*}z - \omega^{*}\omega} \left\langle e^{z\alpha^{*}(t) - z^{*}\alpha(t) + \omega\beta^{*}(t) - \omega^{*}\beta(t)} \right\rangle \;, \tag{77}$$

Now taking into account eqs. (45) and (46) along with their complex conjugates, eq. (77) can be put in the form

$$\Phi_A(z,\omega,t) = e^{-z^{*}z - \omega^{*}\omega + \varepsilon_{11}^{*}z - \varepsilon_{11}z^{*} + \varepsilon_{12}\omega - \varepsilon_{12}^{*}\omega^{*}} \left\langle e^{z\alpha'^{*}(t) - z^{*}\alpha'(t) + \omega\beta'^{*}(t) - \omega^{*}\beta'(t)} \right\rangle \;, \tag{78}$$

where

$$\alpha'(t) = p_1(t)\alpha(0) + q_1(t)\beta^{*}(0) + F_a(t) \;, \tag{79}$$

$$\beta'(t) = p_2(t)\beta^{*}(0) + q_2(t)\alpha(0) + F_b^{*}(t) \;. \tag{80}$$

With the aid of eqs. (47)- (52), it can be easily established that

$$\frac{d}{dt}\langle\alpha'(t)\rangle = -\frac{1}{2}\mu_a\langle\alpha'(t)\rangle + \frac{1}{2}\nu_-\left\langle\beta'^{*}(t)\right\rangle \;, \tag{81}$$

$$\frac{d}{dt}\langle\beta'(t)\rangle = -\frac{1}{2}\mu_c\langle\beta'(t)\rangle + \frac{1}{2}\nu_+\left\langle\alpha'^{*}(t)\right\rangle \;, \tag{82}$$

We see that eqs. (81) and (82) are linear differential equations for $\alpha'(t)$ and $\beta'(t)$. On the other hand, taking into account eqs. (51), (52), and the assumption that the cavity modes are initially in a vacuum state, we have

$$\langle\beta'(t)\rangle = \langle\alpha'(t)\rangle = 0 \;, \tag{83}$$

Thus we observe that $\alpha'(t)$ and $\beta'(t)$ are Gaussian variables with a vanishing mean. In view of this, eq. (78) can be expressed as [20]

$$\Phi_A(z,\omega,t) = e^{-z^{*}z - \omega^{*}\omega + \varepsilon_{11}^{*}z - \varepsilon_{11}z^{*} + \varepsilon_{12}\omega - \varepsilon_{12}^{*}\omega^{*}} \times exp\left[\left\langle \frac{1}{2}\begin{pmatrix} z\alpha'^{*}(t) - z^{*}\alpha'(t) \\ +\omega\beta'^{*}(t) - \omega^{*}\beta'(t) \end{pmatrix}^{2} \right\rangle\right], \tag{84}$$

Hence on account of eqs. (64- 67), the characteristic function can be put in the form

$$\Phi_A(z,\omega,t) = e^{-a_\alpha z^{*}z + z^{*}(\omega^{*}b - \varepsilon_{11}) + z(\omega b^{*} + \varepsilon_{11}^{*})}\; e^{-a_\beta\omega^{*}\omega + \varepsilon_{12}\omega - \varepsilon_{12}^{*}\omega^{*}} \;, \tag{85}$$

where

$$a_\alpha = 1 + \frac{A(1-\eta)(4\kappa + 3A\eta + A) + 16\chi_2^{2}(\kappa + A\eta)}{4\left[\kappa(\kappa + A\eta) - 4\chi_2^{2}\right](2\kappa + A\eta)} \;, \tag{86}$$

$$a_\beta = 1 + \frac{\kappa\left(4\chi_2 + A\sqrt{1-\eta^{2}}\right)^{2}}{4\left[\kappa(\kappa + A\eta) - 4\chi_2^{2}\right](2\kappa + A\eta)} \;, \tag{87}$$

and

$$b=\frac{\kappa\left(4\chi_2+A\sqrt{1-\eta^2}\right)(2\kappa+A\eta+A)}{4\left[\kappa(\kappa+A\eta)-4\chi_2^2\right](2\kappa+A\eta)} \quad , \tag{88}$$

Now introducing (85) into eq (74) and carrying out the integration with the help of

$$\int\frac{d^2z}{\pi^2}\quad exp(-azz^*+bz+cz^*+Az^2+Bz^{*2})$$
$$=\frac{1}{\sqrt{a^2-4AB}}exp\left[\frac{abc+Ac^2+Bb^2}{a^2-4AB}\right] \quad , \quad a>0 \tag{89}$$

we obtain

$$Q(\alpha,\beta,t)=\frac{u_\alpha u_\beta-\upsilon^*\upsilon}{\pi^2}\quad exp[-u_\beta\alpha^*\alpha+$$
$$\alpha\left(p^*+\upsilon^*\beta\right)+\alpha^*(p+\upsilon\beta^*)$$
$$-u_\alpha\beta^*\beta+\beta q^*+\beta^*q-\varepsilon_{11}p^*-\varepsilon_{12}q] \quad , \tag{90}$$

where

$$u_\alpha=\frac{a_\alpha}{a_\alpha a_\beta-b^*b} \quad , \tag{91}$$

$$u_\beta=\frac{a_\beta}{a_\alpha a_\beta-b^*b} \quad , \tag{92}$$

$$\upsilon=\frac{b}{a_\alpha a_\beta-b^*b} \quad , \tag{93}$$

$$p=u_\beta\varepsilon_{11}-\upsilon\varepsilon_{12} \quad , \tag{94}$$

$$q=u_\alpha\varepsilon_{12}^*-\upsilon\varepsilon_{11}^* \quad . \tag{95}$$

6. Mean of the photon number sum and difference

We next proceed to calculate the mean and variances of the photon number sum and difference of mode *a* and mode *b* applying the Q function. We define the operators representing the photon number sum and difference of mode *a* and mode *b* by

$$\hat{n}_\pm=\hat{a}^\dagger\hat{a}\pm\hat{b}^\dagger\hat{b} \quad , \tag{96}$$

The mean of the photon number sum and difference can be written in terms of the Q function as
Z

$$\bar{n}_\pm=\int d^2\alpha\ \ d^2\beta\ \ Q(\alpha,\beta,t)\ \ \overline{\left(\alpha^*\alpha\pm\beta^*\beta-1\mp1\right)} \quad , \tag{97}$$

Now applying the Q function eq. (90) in eq. (97) and performing the integration with the help of eq. (89), we obtain

$$\bar{n}_\pm=e^{-\varepsilon_{11}p^*-\varepsilon_{12}q}\left(\frac{\partial^2}{\partial p^*\partial p}\pm\frac{\partial^2}{\partial q^*\partial q}-1\mp1\right)$$
$$exp\left\{\frac{u_\alpha p^*p+u_\beta q^*q-\upsilon^*pq+\upsilon p^*q^*}{u_\alpha u_\beta-\upsilon^*\upsilon}\right\} \quad , \tag{98}$$

from which follows

$$\bar{n}_\pm=\bar{n}_a\pm\bar{n}_b \quad , \tag{99}$$

where

$$\bar{n}_a=\frac{u_\alpha}{u_\alpha u_\beta-\upsilon^*\upsilon}+\varepsilon_{11}^*\varepsilon_{11}-1, \tag{100}$$

and

$$\bar{n}_b=\frac{u_\beta}{u_\alpha u_\beta-\upsilon^*\upsilon}+\varepsilon_{12}^*\varepsilon_{12}-1, \tag{101}$$

are the mean photon numbers of mode *a* and mode *b*. With the aid of eqs. (91), (92), (86), and (87), we can write

$$\bar{n}_a=\frac{4A(1-\eta)(4\kappa+3A\eta+A)+16\chi_2^2(\kappa+A\eta)}{4\left[\kappa(2\kappa+A\eta)-4\chi_2^2\right](2\kappa+A\eta)}$$
$$+\frac{\chi_1^2\left(2\kappa+A\eta+A+4\chi_2-A\sqrt{1-\eta^2}\right)^2}{\left[\kappa(\kappa+A\eta)-4\chi_2^2\right]^2}, \tag{102}$$

and

$$\bar{n}_b=\frac{\kappa\left(4\chi_2+A\sqrt{1-\eta^2}\right)^2}{4\left[\kappa(2\kappa+A\eta)-4\chi_2^2\right](2\kappa+A\eta)}$$
$$+\frac{\chi_1^2\left(2\kappa+A\eta-A+4\chi_2+A\sqrt{1-\eta^2}\right)^2}{\left[\kappa(\kappa+A\eta)-4\chi_2^2\right]^2} \quad , \tag{103}$$

We easily see from eqs. (102) and (103) that the driving coherent light enhances the mean photon numbers of mode *a* and mode *b*. On account of eqs. (102) and (103), the mean of the photon number sum and difference can be written as

$$\bar{n}_\pm=\frac{\begin{array}{c}2\kappa A(1-\eta)(2\kappa+A\eta)+16\chi_2^2A\eta\pm8\kappa\chi_2A\sqrt{1-\eta^2}\\+(1\pm1)\kappa\left[A^2(1-\eta^2)+16\chi_2^2\right]\end{array}}{4\left[\kappa(\kappa+A\eta)-4\chi_2^2\right](2\kappa+A\eta)}$$
$$+\frac{\chi_1^2\left[\begin{array}{c}\left(2\kappa+A\eta+A+4\chi_2-A\sqrt{1-\eta^2}\right)^2\pm\\\left(2\kappa+A\eta-A+4\chi_2+A\sqrt{1-\eta^2}\right)^2\end{array}\right]}{\left[\kappa(\kappa+A\eta)-4\chi_2^2\right]^2} \quad , \tag{104}$$

We now proceed to consider some special cases. We first consider the case in which the parametric amplifier and the driving coherent light are absent. Thus upon setting $\chi_1=\chi_2=0$ in eq. (104), we get

$$\bar{n}_\pm=A(1-\eta)\ \ \frac{2(2\kappa+A\eta)+(1\pm1)A(1+\eta)}{4(\kappa+A\eta)(2\kappa+A\eta)} \quad , \tag{105}$$

This is the mean of the photon number sum and difference for the cavity modes produced by two different nondegenerate three-level atom coupled to a vacuum reservoir. We see from eq. (105) that the mean of the photon number difference is positive. This shows that the mean photon number of mode *a* is greater than that of mode *b*. We observe from Figure. 5 that the mean photon

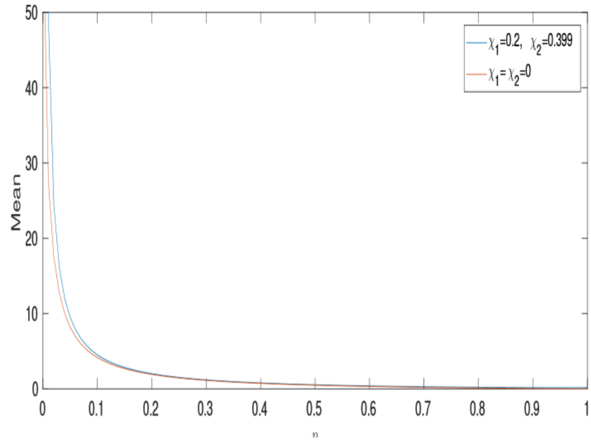

Figure 5. Plots of the mean of the photon number difference [eq. (104)] versus η (blue curve) and the variances of the photon number difference [eq. (105)] versus η (red curve) for $A = 100$, $\kappa = 0.8$ $\chi_1 = 0.2$, and $\chi_2 = 0.399$.

number increases considerably due to the driving coherent light and parametric amplifier. We next consider the case in which atoms are not injected into the cavity. Hence upon setting $A = 0$ in eq. (104), we find

$$\bar{n}_{\pm} = (1 \pm 1)\left[\frac{2\chi_2^2}{\kappa^2 - 4\chi_2^2} + \frac{4\kappa^2\chi_1^2}{\left(\kappa^2 - 4\chi_2^2\right)^2}\right], \qquad (106)$$

This represents the mean of the photon number sum and difference of the cavity modes for a two nondegenerate parametric oscillator driven by coherent light and coupled to a vacuum reservoir. We see from eq. (106) that the mean of the photon number difference is zero. We observe from these two special cases that the mean photon number of mode a is greater than that of mode b due to the three-level laser. And the increase in the mean photon number of mode a must be due to the decay of some atoms from the intermediate level to levels other than level c spontaneously.

7. Variances of the photon number sum and difference

The variances of the photon number sum and difference defined by

$$\Delta n_{\pm}^2 = \left\langle \left(\hat{a}^{\dagger}\hat{a} \pm \hat{b}^{\dagger}\hat{b}\right)^2 \right\rangle - \left\langle \hat{a}^{\dagger}\hat{a} \pm \hat{b}^{\dagger}\hat{b}\right\rangle^2, \qquad (107)$$

can be expressed as

$$\Delta n_{\pm}^2 = \Delta n_a^2 + \Delta n_b^2 \pm 2n_{ab}, \qquad (108)$$

in which

$$\Delta n_a^2 = \left\langle \left(\hat{a}^{\dagger}\hat{a}\right)^2 \right\rangle - \bar{n}_a^2, \qquad (109)$$

is the photon number variance of mode a,

$$\Delta n_b^2 = \left\langle \left(\hat{b}^{\dagger}\hat{b}\right)^2 \right\rangle - \bar{n}_b^2, \qquad (110)$$

is the photon number variance of mode b, and

$$n_{ab} = \left\langle \hat{a}^{\dagger}\hat{a}\hat{b}^{\dagger}\hat{b}\right\rangle - \bar{n}_a\bar{n}_b, \qquad (111)$$

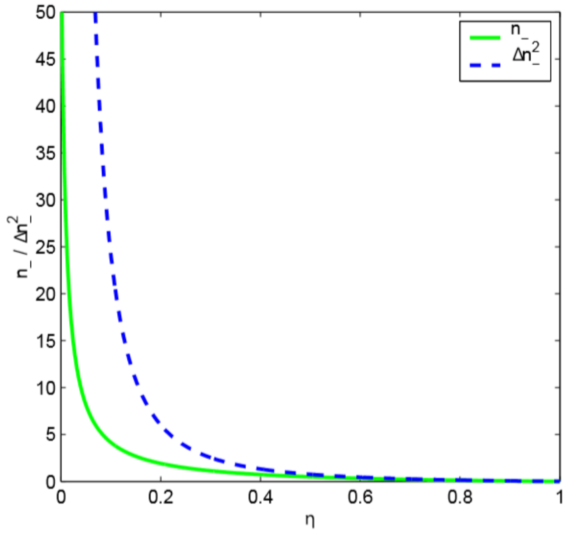

Figure 6. Plots of the mean of the photon number difference [eq. (105)] versus η (solid curve) and the variances of the photon number difference [eq. (119)] versus η (dashed curve) for $A = 100$, $\kappa = 0.8$.

with $\bar{n}_a = \langle \hat{a}^{\dagger}\hat{a}\rangle$ and $\bar{n}_b = \langle \hat{b}^{\dagger}\hat{b}\rangle$. Using the commutation relation $[\hat{a}, \hat{a}^{\dagger}] = 1$, we can write

$$\Delta n_a^2 = \left\langle \hat{a}^2\hat{a}^{\dagger 2}\right\rangle - \bar{n}_a^2 - 3\bar{n}_a - 2, \qquad (112)$$

The first term on the right side of eq. (112) can be expressed in terms of the Q function as [3]

$$\left\langle \hat{a}^2\hat{a}^{\dagger 2}\right\rangle = \int d\alpha^2\, d\beta^2\, Q(\alpha,\beta,t)\ \alpha^{*2}\alpha^2, \qquad (113)$$

Now applying the Q function eq. (90) in eq. (113) and performing the integration, we obtain

$$\left\langle \hat{a}^2\hat{a}^{\dagger 2}\right\rangle = 2(\bar{n}_a + 1)^2 - |\varepsilon_{11}|^4, \qquad (114)$$

Therefore, substitution of eq. (114) into eq. (112) yields

$$\Delta n_a^2 = \bar{n}_a^2 + \bar{n}_a - |\varepsilon_{11}|^4, \qquad (115)$$

Following the same procedure, we easily obtain

$$\Delta n_b^2 = \bar{n}_b^2 + \bar{n}_b - |\varepsilon_{12}|^4, \qquad (116)$$

and

$$n_{ab} = \left|b + \varepsilon_{11}\varepsilon_{12}^*\right|^2 - |\varepsilon_{11}|^2\left|\varepsilon_{12}^*\right|^2, \qquad (117)$$

Hence combination of eqs. (108), (115), (116), and (117) results in

$$\Delta n_{\pm}^2 = \bar{n}_a^2 + \bar{n}_a + \bar{n}_b^2 + \bar{n}_b \pm 2\left|b + \varepsilon_{11}\varepsilon_{12}^*\right|^2 - \left(|\varepsilon_{11}|^2 \pm \left|\varepsilon_{12}^*\right|^2\right)^2, \qquad (118)$$

Upon setting $\chi_1 = \chi_2 = 0$ in eq. (118), we get

$$\Delta n_{\pm}^2 = \bar{n}_a^2 + \bar{n}_a + \bar{n}_b^2 + \bar{n}_b \pm 2|b|^2, \qquad (119)$$

We observe from figure 6 that the variance of the photon number difference is greater than the mean of the photon number difference.

8. Conclusion

In this paper, we have studied the squeezing and statistical properties of the cavity modes produced by **two**

nondegenerate three-level lasers whose cavities contain a parametric amplifier, with the cavity modes driven by coherent light and coupled to a vacuum reservoir. We have obtained, using the master equation, stochastic differential equations associated with the normal ordering. Applying the solutions of the resulting differential equations, we have calculated the quadrature variances. The light produced by the two nondegenerate systems is in a squeezed state. It is found that the parametric amplifier increases the degree of squeezing, but the driving coherent light does not have any effect on the squeezing. We have also seen that the degree of squeezing increases with the linear gain coefficient for small values of η, and almost perfect squeezing can be obtained for large values of the linear gain coefficient. In addition, we have determined, employing the Q function, the mean photon number and the variance of the photon number for the cavity modes. The mean photon number increases considerably due to the driving coherent light and **the** parametric amplifier. Since the effect of the parametric amplifier on the three-level laser is to enhance both the degree of squeezing and the mean photon number, a bright and highly squeezed light can be produced by the quantum optical system considered in this paper.

References

1. E Alebachew and K Fesseha, *arXiv.0506178v2[quan-Ph]* (2005).
2. S Tesfa, *Phys. Rev. A* **74** (2006) 043816.
3. F Kassahun, "*Fundamentals of Quantum Optics*" Lulu Press Inc., North Carolina (2010).
4. T Y Darge and F Kassahun, *Pmc Phys.* B *3* (2010) 1.
5. K Fesseha, *Phys. Rev.* A **63** (2001) 033811.
6. T Abebe, S Mosissa, and N Belay, *Bulg. J. Phys*. **46** , 3 (2019) 214.
7. T Abebe and N Gemechu, *Ukr. J. Phys*. **63**, 7 (2018) 600.
8. S Tesfa, Driven degenerate three-level cascade laser. *arXiv preprint arXiv:0708,* (2007) 2815.
9. E Alebachew, *Opt. Commun*. **280** (2007) 133.
10. E Alebachew and K Fesseha, *Opt. Commun*. **265** (2006) 314.
11. E Alebachew, *Phys. Rev.* A **76** (2007) 023808.
12. T Abebe, *J. Phys.* **63** (2018) 733.
13. B Teklu, *Opt. Commun*. **261** (2006) 310.
14. B Daniel and K Fesseha, *Opt. Commun.* **151** (1998) 384.
15. K Fesseha, *Opt. Commun*. **156** (1998) 145.
16. S Tesfa, *Eur. Phys. J.* D **46** (2008) 351.
17. E Mosisa, *Adv. Math. Phys*. **2021** (2021) 6625690.
18. S M Barnett and P M Radmore, "*Methods in Theoretical Quantum Optics*"Oxford University Press, New York (1997).
19. S Tesfa, *Phys. Rev.* A **77** (2008) 013815.
20. K Fesseha, "Refined Quantum Analysis of Light" Revised Edition (CreateSpace Independent Publishing Platform (2016).
21. A Menisha, *Russ. Laser Res.* **42** (2022) 260.