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# Comparison of tensor and vector theories of gravitation

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**Abstract**
Physical quantities in continuously distributed matter in curved spacetime, and equations for matter and fields are considered both from the point of view of tensor theory of gravitation and on the basis of vector theory of gravitation. An example in the first case is the general theory of relativity (GTR), which uses a scalar pressure field and a scalar acceleration field. In the second case, relativistic vector fields are taken into account, including the covariant theory of gravitation, the pressure vector field and the acceleration vector field. To analyze and compare the results in each approach, formulas derived from the principle of least action and from the corresponding Lagrangian are used. The problem of correlating scalar pressure with the principle of least action in the general relativity is considered. The conclusion is drawn that results of the general relativity, when scalar pressure is taken into account, are valid for relativistic uniform systems, but in a more general case, they require correction. Three versions of general relativity were analyzed: $GTR^1$, $GTR^2$ and $GTR^m$. The $GTR^1$ version is the closest version to the standard general theory of relativity, the $GTR^2$ version follows exactly the principle of least action, and the $GTR^m$ version is characterized by the fact that the acceleration field and pressure field are represented not as scalar fields but as vector fields. Equations for metric, equations of motion, equations for fields, formulas for the energy and momentum, which follow from the Lagrangian formalism, are presented for all versions of general relativity. An explanation is given of where dark energy comes from and what it is whithin general relativity.

**Keywords:** Lagrangian formalism; integral of motion; vector field; general theory of relativity; covariant theory of gravitation.

## 1. Introduction

The general relativity theory is one of the most developed tensor theories of gravitation. In the general relativity, the metric tensor is considered as a characteristic of a special metric field that completely describes gravitational field. Thus, the properties of the gravitational field and its action are reduced to the geometry of spacetime and the metric field. If in a physical system it is necessary to take into account the action of some other field, then this other field must make its contribution to the metric tensor and to the metric field. Each subsequent field changes the metric tensor and through it changes the observed action of other fields in the system. Thus, it turns out that all fields influence each other through the metric.

Due to inclusion of gravitational field in the metric field, a feature appears in general theory of relativity in how the principle of general covariance and the principle of correspondence are understood. The principle of general covariance implies that physical equations should be written in such a way that the form of these equations does not depend on the choice of reference frame and on the choice of coordinate system. According to the correspondence principle, covariantly written equations in a gravitational field tending to zero should transform into corresponding equations of special theory of relativity. In most situations, it is the gravitational field that makes the maximum contribution to the curvature of spacetime. More generally, and especially in alternative theories of gravitation, where gravitation is determined independently of the metric and several different fields act simultaneously, the correspondence principle should be formulated as follows: in weak fields that make a negligible contribution to the curvature of spacetime, covariantly written equations should pass into the corresponding equations of special relativity.

As a rule, the principle of general covariance is fulfilled if the equations are written in terms of invariant scalar functions, four-vectors and four-tensors. To fulfill conditions of the correspondence principle, the total mass of particles of a system is reduced and the system removed from the sources of external gravitational fields so that non-gravitational forces prevail and behave in the same way as in special theory of relativity. In this case, the effects of spacetime curvature become insignificant and gravitational phenomena in moving systems must comply with Newton's law, taking into account the Lorentz transformations for gravitational force.

Let there be covariant equations of some small-sized physical system, and a reference frame is chosen in which gravitational phenomena in the system disappear. In general relativity, this situation leads to the principle of equivalence of gravitational forces and inertial forces, and to the equality of gravitational mass and inertial mass for point-sized bodies.

However, in general case, the equivalence principle cannot be considered as a single general principle for every theory of gravitation, and especially in the case when large bodies are considered. In large bodies, the gravitational acceleration is different at each point and directed in different directions. Therefore, locally inertial reference frames, which can be represented at each point of the body, will be accelerated relative to each other in different directions. This means that for each other, locally inertial reference frames are not inertial systems and cannot be connected to each other by Lorentz transformations. This situation does not meet the conditions of the correspondence principle.

Since it is impossible to reduce the masses of large systems, significant gravitational fields remain in such systems. Therefore, the equivalence principle holds only locally, in small regions of spacetime, but not for the entire system as a whole. Therefore, the equality of gravitational and inertial mass postulated for point bodies cannot be considered accurate for large massive bodies. In this case, the inertial mass of a system of interacting particles is determined by the internal properties of the system and is measured through acceleration of the center of momentum of the system at a given force. The gravitational mass of the system is found in another way, through interaction of the system with a small test body of known mass located at some distance from the system. Due to difference in definition of these masses, the supposed equality of gravitational and inertial masses of a system, as well as definition of the inertial mass of the system itself, remains a subject of discussion in general relativity.

Unlike the general relativity, in covariant theory of gravitation, which is a vector theory, gravitational interaction is not completely reduced to the curvature of spacetime. Moreover, even in ideal case, in flat Minkowski spacetime, when the metric tensor does not depend on time and coordinates, the gravitational force is assumed to be the same independently existing physical force as the electromagnetic force.

The covariant theory of gravitation proceeds from the four-potential $D_\nu = \left( \frac{\psi}{c}, -\mathbf{D} \right)$, which is described in terms of the scalar potential $\psi$ and the vector potential $\mathbf{D}$ of gravitational field. The gravitational field tensor $\Phi_{\mu\nu}$ is found using the four-curl $\nabla_\mu \times D_\nu = \nabla_\mu D_\nu - \nabla_\nu D_\mu = \Phi_{\mu\nu}$, which allows us to determine the stress-energy tensor of gravitational field, including the energy and energy flux of the field [1].

Instead, in general relativity, we proceed not from the four-vector in the form of the four-potential, but from the four-tensor, namely, from the metric tensor $g_{\mu\nu}$. The Christoffel symbols and curvature tensor are expressed in terms of derivatives of the metric tensor with respect to coordinates and time, with the help of which all the gravitational effects are then specified.

In the case of a continuous distribution of matter, in many works in general relativity difficulties arise due to the fact that either a non-covariant Lagrangian is used, or non-four-dimensional coordinates and momenta are used. In order to overcome these difficulties and express the Hamiltonian in covariant form, it is proposed in [2] to use the DeDonder-Weyl formalism. In this case, four additional axioms are taken into account.

Analysis of general relativity and comparison of it with the theory of vector fields leads to the following. Direct inclusion of the scalar pressure $\mathbb{P}_0$ into Lagrangian density in general relativity is difficult, since there is no direct relationship between the variation $\delta \mathbb{P}_0$ and other variables. In this regard, there is no standard expression for the Lagrangian density in the general relativity, from which covariant expressions follow both for $\mathbb{P}_0$ in four-dimensional form, and for the stress-energy tensor in continuous matter. Instead, various forms of such Lagrangian densities have been proposed [3-5].

Unfortunately, the relationships between gravitation and geometry, as well as reduction of physics to mathematics, create additional significant problems in general relativity. Among the latest works aimed at solving these problems, one can point to article [6], which analyses methods for determining the energy and momentum of gravitational field. An attempt is made to explain the problem of cosmological constant and find the law of conservation of the energy-momentum in general relativity. In [7], the energy and momentum of a star were estimated, using the model of matter as an ideal fluid in which a scalar pressure field acts.

The main drawback of general relativity is that energy and momentum of a system are usually not expressed by standard formulas of Lagrangian formalism, but rather volume integration of time components of stress-energy tensor summed up with gravitational pseudotensor components. It is believed that a four-dimensional quantity (integral pseudovector) obtained in this way makes it possible to find four-momentum of a system. However, if one calculates an integral pseudovector in the theory of vector fields, it turns out that such a pseudovector describes distribution of energy and energy fluxes of fields of the system and is not a four-vector [8]. Indeed, in a closed stationary system with a constant metric not only the energy, momentum and angular momentum are conserved, but also configuration of spatial distribution of the field energy. Moreover, the general relativity includes at least 7 different forms of gravitational pseudotensors [9], which leads to different integral pseudovectors with noncoincident spatial distributions of the fields' energy and to the problem of interpreting an integral pseudovector as a uniquely defined integral of motion.

When building cosmological models in general relativity, we are faced with a number of problems associated with cosmological constant, singularities, and anomalies of cosmic microwave background radiation, as well as with the need to introduce concepts such as dark matter, and dark energy. To solve these problems, such works appear, in which, among other things, vector-tensor theories of gravitation are considered [10-16], and the prospects of these theories for future research are shown. This may also apply to covariant theory of gravitation, which is a vector theory. In particular, in [17] the metric outside a massive body was calculated, which characterizes spacetime within the framework of covariant theory of gravitation, and in [18] the metric inside the body was found. Similar calculations can be used to determine the metric in cosmology. In covariant theory of gravitation, the Pioneer effect is explained, which should not exist according to general relativity [19].

A physical system, consisting of particles with the same charge-to-mass ratio, cannot radiate in a dipole manner. The same applies to radiation of gravitational waves by a system of neutral massive particles. In covariant theory of gravitation, gravitational dipole radiation is possible from any accelerated mass, however, the total dipole radiation from a closed physical system is close to zero due to mutual cancellation of oppositely directed radiation from the system's parts. The quadrupole radiation remains the same as is in the general relativity. Thus, both the covariant theory of gravitation and the general relativity predict quadrupole-type gravitational waves from massive cosmic objects; these waves were recently discovered and presented in [20-21].

As a rule, when calculating using the general relativity, pressure in matter is considered as a scalar field. In the simplest case of stationary matter, it is assumed that the scalar isotropic pressure $\mathbb{P}_0$ does not influence the energy density in time component of stress-energy tensor of matter. In contrast, when calculating using covariant theory of gravitation, pressure is considered as a vector field, so that the energy density turns out to be dependent on the scalar potential of pressure field [22]. A similar situation arises in relation to acceleration field, which in general relativity is represented as a scalar field. Thus, the Lagrangian of general relativity with scalar fields in matter differs significantly from the Lagrangian for vector fields and covariant theory of gravitation.

In [23] it was shown how vector fields are combined into a single general field. In the concept of vector fields, it was possible to find formulas for kinetic energy and for distribution of particle velocities inside a relativistic uniform system [24], as well as to derive the generalized virial integral theorem [25], the Navier–Stokes equation [22], the equations of motion of matter particles [26], expressions for covariant additive integrals of motion [27], derive the generalized Poynting theorem and give a solution to the 4/3 problem [28], estimate the parameters of planets and stars [29], prove the integral field theorem [30], find the generalized four-momentum [31] and four-momentum of a physical system [8] in curved space-time in continuously distributed matter.

The purpose of this work is to use the Lagrangian formalism [32] to analyze the general relativity, indicating the difficulties that arise from the point of view of theoretical approach. In particular, the well-known problem of general relativity with determining the mass, energy and momentum of a system in gravitational field is solved by using auxiliary quantities that represent the gravitational field as a vector field.

The principle of least action makes it possible to study physical systems and find equations of motion not only in the Lagrangian, but also in the Hamiltonian formulation [33], [34]. However, the Lagrangian formulation is considered more fundamental [35], while the well-known Lagrangian for vector fields is not difficult to adapt for general relativity. This makes it quite easy to compare the results obtained in general relativity and in the theory of vector fields.

In our calculations, we will everywhere use the metric signature of the form (+,–,–,–).

## 2. Methods

Let us consider the particulars of application of Lagrangian formalism in general relativity. Having studied a great number of papers, we have not ever found a Lagrangian density, which allows us to uniquely express the scalar isotropic pressure in a four-dimensional form while providing the standard stress-energy tensor of the general relativity for continuous matter. As a result, we had to construct such a Lagrangian density $\mathcal{L} = \mathcal{L}_p + \mathcal{L}_f$ by ourselves, which consisted of two parts

$$\begin{aligned}\mathcal{L}_p &= -A_\mu\, j^\mu - c\sqrt{J^\mu J^\nu g_{\mu\nu}} \\ &-K(J^\mu, g_{\mu\nu}) = \\ &= -\frac{1}{c}\rho_{0q}u^0\varphi + \frac{1}{c}\rho_{0q}u^0\mathbf{A}\cdot\mathbf{v} - \\ &c\sqrt{J^\mu J^\nu g_{\mu\nu}} - K(J^\mu, g_{\mu\nu}).\end{aligned} \tag{1}$$

$$\begin{aligned}\mathcal{L}_f &= -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} + c\,k\,R - 2c\,k\,\Lambda = \\ &-\frac{1}{4\mu_0}F_{\mu\nu}F_{\eta\lambda}g^{\mu\eta}g^{\lambda\nu} + c\,k\,R_{\mu\nu}g^{\mu\nu} - 2c\,k\,\Lambda\end{aligned}, \tag{2}$$

where $A_\mu = \left(\frac{\varphi}{c}, -\mathbf{A}\right)$ is the four-potential of electromagnetic field, given by the scalar potential $\varphi$ and the vector potential $\mathbf{A}$ of this field, $j^\mu = \rho_{0q}u^\mu$ is the charge four-current, $\rho_{0q}$ is invariant charge density in the particle's comoving reference frame, $u^{\mu}$ is the four-velocity of a point particle or element of matter, $u^0$ is time component of four-velocity, $\mathbf{v}$ is three-dimensional velocity of a particle or element of matter, $c$ is the speed of light, $J^{\mu}=\rho_0 u^{\mu}$ is the mass four-current, $\rho_0$ is invariant mass density in the particle's comoving reference frame, $K(J^{\mu},g_{\mu\nu})$ is scalar function depending on the four-current $J^{\mu}$ and the metric tensor $g_{\mu\nu}$, $\mu_0$ is the magnetic constant, $F_{\mu\nu}=\nabla_{\mu}A_{\nu}-\nabla_{\nu}A_{\mu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$ is the electromagnetic tensor, $k=-\frac{c^3}{16\pi G\beta}=-\frac{1}{2c\varkappa}$, where $\varkappa$ is Einstein's gravitational constant, $R_{\mu\nu}$ is the Ricci tensor, $R$ is the scalar curvature, $\Lambda$ is the cosmological constant.

The Lagrangian density $\mathcal{L}=\mathcal{L}_p+\mathcal{L}_f$ with components $\mathcal{L}_p$ (1) and $\mathcal{L}_f$ (2) has a slight difference from the standard Lagrangian density of general relativity in [36-38], taking into account matter and electromagnetic field. This difference lies only in the fact that a scalar function $K(J^{\mu},g_{\mu\nu})$ is introduced in (1). Given the function $K(J^{\mu},g_{\mu\nu})$, based on the principle of least action, those terms appear in equation for the metric and in equation of motion of matter that can be associated with scalar pressure $\mathbb{P}_0$ and with the pressure force in matter.

The four-currents $j^{\mu}$ and $J^{\mu}$ in (1) are four -vectors, as defined in [36-37], where the variations of these four - currents, necessary in the principle of least action, are also calculated. In this case, the continuity equations have the form $\nabla_{\mu} j^{\mu}=0$, $\nabla_{\mu} J^{\mu}=0$. Covariant expressions for four-currents $j^{\mu}=\rho_{0q}u^{\mu}$ and $J^{\mu}=\rho_0 u^{\mu}$ correspond exactly to four-vector algebra, since they are obtained by multiplying the invariant scalars $\rho_{0q}$ and $\rho_0$ by the four-velocity $u^{\mu}$.

A feature of $\mathcal{L}_p$ (1) is the direct dependence on the four-currents $j^{\mu}$ and $J^{\mu}$, whereas in $\mathcal{L}_f$ (2) there is no such dependence. Note that the term $-c\sqrt{J^{\mu}J^{\nu}g_{\mu\nu}}$ in (1), when integrated over the invariant four-volume in the action function and with subsequent variation, gives the same result in the principle of least action as the corresponding term $-\left(p^{\mu}p_{\mu}\right)^{1/2}$ in [37].

In (1). it is essential that the mass four-current $J^{\mu}$ in the radical $\sqrt{J^{\mu}J^{\nu}g_{\mu\nu}}$ should always be used in the form of a contravariant four-vector, and the metric tensor should be taken as a doubly covariant tensor $g_{\mu\nu}$. It is due to this choice that the stress-energy tensor of matter in the general relativity is obtained with a positive sign.

Next, we will need the Lagrangian density $\mathcal{L}=\mathcal{L}_p+\mathcal{L}_f$ for four vector fields according to [1] and [22]

$$\mathcal{L}_p=-A_{\mu}j^{\mu}-D_{\mu}J^{\mu}-U_{\mu}J^{\mu}-\pi_{\mu}J^{\mu}= =\frac{u^0}{c}\begin{pmatrix}-\rho_{0q}\varphi+\rho_{0q}\mathbf{A}\cdot\mathbf{v}-\rho_0\psi+\\ \rho_0\mathbf{D}\cdot\mathbf{v}-\rho_0\vartheta+\rho_0\mathbf{U}\cdot\mathbf{v}-\\ \rho_0\wp+\rho_0\mathbf{\Pi}\cdot\mathbf{v}\end{pmatrix}.$$

$$\mathcal{L}_f=-\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}+\frac{c^2}{16\pi G}\Phi_{\mu\nu}\Phi^{\mu\nu}- \frac{c^2}{16\pi\eta}u_{\mu\nu}u^{\mu\nu}-\frac{c^2}{16\pi\sigma}f_{\mu\nu}f^{\mu\nu}+ ckR-2ck\Lambda, \qquad (3)$$

where $D_{\mu}=\left(\frac{\psi}{c},-\mathbf{D}\right)$ is the four-potential of gravitational field, described in terms of the scalar potential $\psi$ and the vector potential $\mathbf{D}$ within the framework of covariant theory of gravitation, $U_{\mu}=\left(\frac{\vartheta}{c},-\mathbf{U}\right)$ is the four-potential of acceleration field, where $\vartheta$ and $\mathbf{U}$ denote the scalar and vector potentials, respectively, $\pi_{\mu}=\left(\frac{\wp}{c},-\mathbf{\Pi}\right)$ is the four-potential of pressure field, consisting of the scalar potential $\wp$ and the vector potential $\mathbf{\Pi}$, $G$ is the gravitational constant, $\Phi_{\mu\nu}=\nabla_{\mu}D_{\nu}-\nabla_{\nu}D_{\mu}=\partial_{\mu}D_{\nu}-\partial_{\nu}D_{\mu}$ is the gravitational tensor, $\eta$ is the acceleration field coefficient, $u_{\mu\nu}=\nabla_{\mu}U_{\nu}-\nabla_{\nu}U_{\mu}=\partial_{\mu}U_{\nu}-\partial_{\nu}U_{\mu}$ is the acceleration tensor, calculated as the four-curl of the four-potential of acceleration field, $\sigma$ is the pressure field coefficient, $f_{\mu\nu}=\nabla_{\mu}\pi_{\nu}-\nabla_{\nu}\pi_{\mu}=\partial_{\mu}\pi_{\nu}-\partial_{\nu}\pi_{\mu}$ is the pressure field tensor.

Similar to electromagnetic field equations, the gravitational field equations connect the gravitational tensor $\Phi_{\mu\nu}$ with the mass four-current $J^{\mu}$ and allow one to calculate the components of gravitational tensor [1]. The equation for calculating the four-potential $D_{\mu}$ has the following form [26]

$$\nabla_{\beta}\nabla^{\beta}D_{\mu}=-\frac{4\pi G}{c^2}J_{\mu}-D^{\beta}R_{\beta\mu},$$

where $R_{\beta\mu}$ is the Ricci tensor.

According to (1-2), the electromagnetic field is fully taken into account in the Lagrangian density as a vector field, and the gravitational field manifests itself exclusively through the metric tensor $g_{\mu\nu}$; therefore, it is defined as a tensor field. The acceleration field has the energy density of $c\sqrt{J^{\mu}J^{\nu}g_{\mu\nu}}=\rho_0 c^2=\rho_0 u_{\mu}u^{\mu}=u_{\mu}J^{\mu}$ and is represented as a scalar field. This is evident from the fact that in the Lagrangian density (1-2) there is no additional tensor invariant $-\frac{c^2}{16\pi\eta}u_{\mu\nu}u^{\mu\nu}$, associated with the acceleration field, while in the Lagrangian density for vector fields (3) this tensor invariant is present. One of the reasons that the electromagnetic field is represented in Lagrangian densities (1-3) is that in both the covariant theory of gravity and the general theory of relativity, the terms with the electromagnetic field have the same form. On the other hand, electromagnetic fields are of great importance in the physics of relativistic charged particles and in the astrophysics of stars, especially for white dwarfs and neutron stars. Thus, the results obtained in this work can be useful in the analysis of phenomena involving electromagnetic fields.

Note that the four-velocity $u_{\mu}$ is a special and limiting case of the four-potential $U_{\mu}$ of acceleration field, when each particle is considered as a point solid body moving by inertia. The expression $c\sqrt{J^{\mu}J^{\nu}g_{\mu\nu}}=u_{\mu}J^{\mu}$ in (1) in its meaning corresponds to the term $U_{\mu}J^{\mu}$ in the Lagrangian density (3) for vector fields. Thus. the vector acceleration field in the Lagrangian density (3) for vector fields includes the scalar acceleration field of general relativity in (1) as a special case.

The mass density $\rho_0$, charge density $\rho_{0q}$, and scalar function $K(J^{\mu},g_{\mu\nu})$ in (1) are invariant quantities, since they are given in the reference frame that comoves with the matter element under consideration. This means, for example, that the mass density $\rho_0=\frac{1}{c^2}u_{\mu}J^{\mu}$ is expressed in terms of the tensor invariant and therefore is a scalar function. Although in each reference frame, the four-current $J^{\mu}$ and the four-velocity $u_{\mu}$ of a matter element have their own values, the tensor invariant of these quantities always defines the mass density as equal to the mass density $\rho_0$ in the comoving reference frame. A similar reasoning applies to the function $K(J^{\mu},g_{\mu\nu})$.

Our goal will be to find the equation for the metric in general relativity, to derive the formulas for the energy and momentum, to obtain the equation of motion and to relate the function $K(J^{\mu},g_{\mu\nu})$ with the scalar isotropic pressure $\mathbb{P}_0$ in matter.

First, we consider the equation for the metric. Since four-currents $j^{\mu}$ and $J^{\mu}$, four-potential $A_{\mu}$, and the metric tensor $g_{\mu\nu}$ are independent variables when varied, variation $\mathcal{L}_p$ (1) with respect to the metric tensor $g_{\mu\nu}$ can be written as follows

$$\left(\delta\mathcal{L}_p\right)_{g_{\mu\nu}}=-\frac{cJ^{\mu}J^{\nu}\delta g_{\mu\nu}}{2\sqrt{J^{\mu}J^{\nu}g_{\mu\nu}}}-\frac{\partial K}{\partial g_{\mu\nu}}\delta g_{\mu\nu}= -\frac{1}{2}\rho_0 u^{\mu}u^{\nu}\,\delta g_{\mu\nu}-\frac{\partial K}{\partial g_{\mu\nu}}\delta g_{\mu\nu}. \quad (4)$$

Since in (4) $\delta g_{\mu\nu}=-g_{\alpha\mu}g_{\beta\nu}\delta g^{\alpha\beta}$, the variation $\mathcal{L}_p$ with respect to the metric tensor $g^{\alpha\beta}$ and the corresponding functional derivative are equal to

$$\left(\delta\mathcal{L}_p\right)_{g^{\alpha\beta}}=\frac{1}{2}\rho_0 u_{\alpha}u_{\beta}\,\delta g^{\alpha\beta}+ \frac{\partial K}{\partial g_{\mu\nu}}g_{\alpha\mu}g_{\beta\nu}\delta g^{\alpha\beta}.$$

$$\frac{\partial\mathcal{L}_p}{\partial g^{\mu\nu}}=\frac{1}{2}\rho_0 u_{\mu}u_{\nu}+\frac{\partial K}{\partial g_{\alpha\beta}}g_{\mu\alpha}g_{\nu\beta}. \quad (5)$$

For variation $\mathcal{L}_f$ (2) with respect to the metric tensor $g^{\alpha\beta}$ and for the corresponding functional derivative, we can write similarly [36], [38]

$$\left(\delta\mathcal{L}_f\right)_{g^{\mu\nu}}=\frac{1}{2\mu_0}F_{\nu\alpha}F^{\alpha}_{\ \mu}\delta g^{\mu\nu}+ c\,k\,R_{\mu\nu}\delta g^{\mu\nu}$$

$$\frac{\partial\mathcal{L}_f}{\partial g^{\mu\nu}}=\frac{1}{2\mu_0}F_{\nu\alpha}F^{\alpha}_{\ \mu}+c\,k\,R_{\mu\nu}. \quad (6)$$

Taking (5-6) into account, we find the derivative of the entire Lagrangian density $\mathcal{L}=\mathcal{L}_p+\mathcal{L}_f$ with respect to the metric tensor in the general relativity

$$\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}=\frac{1}{2}\rho_0 u_{\mu}u_{\nu}+\frac{\partial K}{\partial g_{\alpha\beta}}g_{\mu\alpha}g_{\nu\beta}+ \frac{1}{2\mu_0}F_{\nu\alpha}F^{\alpha}_{\ \mu}+c\,k\,R_{\mu\nu}. \quad (7)$$

According to the principle of least action, to find the equations of motion of particles and fields, variation of the action $S=\int_{t_1}^{t_2}L\,dt$ should be equated to zero

$$\delta S = \int_{t_1}^{t_2} \delta L\, dt = \int_{t_1}^{t_2} \left( \int_V \delta\left( \mathcal{L}\sqrt{-g} \right) dx^1 dx^2 dx^3 \right) dt = 0 \quad (8)$$

In (8) there is the Lagrangian $L = \int_V \mathcal{L}\sqrt{-g}\, dx^1 dx^2 dx^3$, found by integrating the Lagrangian density $\mathcal{L}$ over the moving volume of the system. Since $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}$, from (8) follows the expression

$$\delta S = \int_{t_1}^{t_2}\int_V \delta\mathcal{L}\sqrt{-g}\, dx^1 dx^2 dx^3 dt - \frac{1}{2}\int_{t_1}^{t_2}\int_V \mathcal{L}\, g_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}\, dx^1 dx^2 dx^3 dt = 0 \quad (9)$$

The metric tensor $g^{\mu\nu}$ is included in the set of independent variables by which the Lagrangian density is varied. We can assume that the Lagrangian density $\mathcal{L} = \mathcal{L}_p + \mathcal{L}_f$ in (1-2) depends on the following variables

$$\mathcal{L} = \mathcal{L}(j^\mu, J^\mu, A_\mu, F_{\mu\nu}, g^{\mu\nu}). \quad (10)$$

Hence,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial j^\mu}\delta j^\mu + \frac{\partial\mathcal{L}}{\partial J^\mu}\delta J^\mu + \frac{\partial\mathcal{L}}{\partial A_\mu}\delta A_\mu + \frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\delta F_{\mu\nu} + \frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}\delta g^{\mu\nu} \quad (11)$$

Substituting (11) into (9) gives:

$$\delta S = \int_{t_1}^{t_2}\int_V \left( \delta\mathcal{L} - \frac{1}{2}\mathcal{L}\, g_{\mu\nu}\delta g^{\mu\nu} \right) \sqrt{-g}\, dx^1 dx^2 dx^3\, dt =$$

$$= \int_{t_1}^{t_2}\int_V \left[ \begin{array}{l} \frac{\partial\mathcal{L}}{\partial j^\mu}\delta j^\mu + \frac{\partial\mathcal{L}}{\partial J^\mu}\delta J^\mu + \\ \frac{\partial\mathcal{L}}{\partial A_\mu}\delta A_\mu + \frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\delta F_{\mu\nu} + \\ + \left( \frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2}\mathcal{L}\, g_{\mu\nu} \right)\delta g^{\mu\nu} \end{array} \right] \sqrt{-g}\, dx^1 dx^2 dx^3\, dt = 0. \quad (12)$$

The equation for metric follows from the equality to zero the last term in square bracket in (12)

$$\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2}\mathcal{L}\, g_{\mu\nu} = 0. \quad (13)$$

Substituting (1-2) and (7) into (13) gives the following

$$\frac{1}{2}\rho_0 u_\mu u_\nu + \frac{\partial K}{\partial g_{\alpha\beta}} g_{\mu\alpha} g_{\nu\beta} + \frac{1}{2\mu_0} F_{\nu\alpha} F^{\alpha}_{\ \mu} + c\,k\,R_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} \left( \begin{array}{l} A_\beta\, j^\beta + c\sqrt{J^\alpha J^\beta g_{\alpha\beta}} + K + \\ \frac{1}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta} - c\,k\,R + 2c\,k\,\Lambda \end{array} \right). \quad (14)$$

Let us write the standard expression for stress-energy tensor $W_{\mu\nu}$ of electromagnetic field, as well as the expression for stress-energy tensor $\tau_{\mu\nu}$ of matter considering the scalar pressure $\mathbb{P}_0$ which is used in general relativity in the limit of continuous matter:

$$W_{\mu\nu} = \frac{1}{\mu_0}\left( F_{\mu\alpha} F^{\alpha}_{\ \nu} + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right). \quad (15)$$

$$\tau_{\mu\nu} = \left( \rho_0 + \frac{\mathbb{P}_0}{c^2} \right) u_\mu u_\nu - \mathbb{P}_0\, g_{\mu\nu}. \quad (16)$$

Substituting (15-16) into (14) taking into account the equality $c\sqrt{J^\alpha J^\beta g_{\alpha\beta}} = \rho_0 c^2$ gives

$$2c\,k\,R_{\mu\nu} - c\,k\,R\,g_{\mu\nu} = -g_{\mu\nu}\left( A_\beta\, j^\beta + \rho_0 c^2 + \mathbb{P}_0 + K + 2c\,k\,\Lambda \right) - 2\frac{\partial K}{\partial g_{\alpha\beta}} g_{\mu\alpha} g_{\nu\beta} + \frac{\mathbb{P}_0}{c^2} u_\mu u_\nu - \tau_{\mu\nu} - W_{\mu\nu}. \quad (17)$$

We apply contraction of equation (17) by multiplying by $g^{\mu\nu}$ and take into account expression $\tau_{\mu\nu}$ (16), as well as the equalities $g_{\mu\nu} g^{\mu\nu} = 4$, $g^{\mu\nu} u_\mu u_\nu = c^2$, $g^{\mu\nu} W_{\mu\nu} = 0$ :

$$2c\,k\,R = 5\rho_0 c^2 + 4A_\beta\, j^\beta + 4K + 2\frac{\partial K}{\partial g_{\alpha\beta}} g_{\alpha\beta} + 8c\,k\,\Lambda \quad (18)$$

In Sections 3.1 and 3.2 we turn to equations (17-18) in connection with the problem of energy gauging and definition of the meaning of cosmological constant $\Lambda$.

According to [8], the energy of a physical system is written as follows

$$E = \int_V \frac{\partial}{\partial\mathbf{v}}\left( \frac{\mathcal{L}_p}{u^0} \right)\cdot\mathbf{v}\, u^0 \sqrt{-g}\, dx^1 dx^2 dx^3 - \int_V \left( \mathcal{L}_p + \mathcal{L}_f \right)\sqrt{-g}\, dx^1 dx^2 dx^3 + \sum_{n=1}^{N}\left( \mathbf{v}_n \cdot \frac{\partial L_f}{\partial \mathbf{v}_n} \right) \quad (19)$$

In (19) $\mathbf{v}_n$ is the velocity of a particle or element of matter of the system with number $n$, the quantity $L_f = \int_V \mathcal{L}_f \sqrt{-g}\, dx^1 dx^2 dx^3$ is the Lagrangian associated with the Lagrangian density $\mathcal{L}_f$. Substituting $\mathcal{L}_p$ (1) and $\mathcal{L}_f$ (2) into (19), taking into account the equality $c\sqrt{J^\alpha J^\beta g_{\alpha\beta}} = \rho_0 c^2$, we find:

$$E = \int_V \left[ -\frac{\partial}{\partial \mathbf{v}} \left( \frac{1}{c} \rho_{0q} \varphi + \frac{\rho_0 c^2 + K}{u^0} \right) + \frac{1}{c} \frac{\partial(\rho_{0q} \mathbf{A})}{\partial \mathbf{v}} \cdot \mathbf{v} \right] \cdot \mathbf{v}\, u^0 \sqrt{-g}\, dx^1 dx^2 dx^3 + \int_V \left( \frac{1}{c} \rho_{0q} u^0 \varphi + \rho_0 c^2 + K + \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - c\, k\, R + 2c\, k\, \Lambda \right) \sqrt{-g}\, dx^1 dx^2 dx^3 + \sum_{n=1}^{N} \left( \mathbf{v}_n \cdot \frac{\partial L_f}{\partial \mathbf{v}_n} \right). \quad (20)$$

If the electromagnetic field potentials $\varphi$ and $\mathbf{A}$ depend on the velocity, then the terms with partial derivatives $\frac{\partial}{\partial \mathbf{v}}$ in (20) will not be equal to zero. In some cases, we can assume that the sum $\rho_0 c^2 + K$ does not directly depend on the particle velocity $\mathbf{v}$; however, the time component $u^0$ of the particles' four-velocity in the general case depends on the velocity $\mathbf{v}$. Indeed, in the limit of the special theory of relativity $u^0 = \gamma c = \frac{c}{\sqrt{1 - \mathbf{v}^2/c^2}}$.

According to [8], the relativistic momentum of a system is expressed by the formula:

$$\mathbf{P} = \int_V \frac{\partial}{\partial \mathbf{v}} \left( \frac{\mathcal{L}_p}{u^0} \right) u^0 \sqrt{-g}\, dx^1 dx^2 dx^3 + \sum_{n=1}^{N} \frac{\partial L_f}{\partial \mathbf{v}_n}. \quad (21)$$

Substitution into (21) $\mathcal{L}_p$ (1) and $\mathcal{L}_f$ (2) taking into account the equality $c\sqrt{J^\alpha J^\beta g_{\alpha\beta}} = \rho_0 c^2$ gives

$$\mathbf{P} = \int_V \left[ \frac{1}{c} \rho_{0q} \mathbf{A} - \frac{\partial}{\partial \mathbf{v}} \left( \frac{1}{c} \rho_{0q} \varphi + \frac{\rho_0 c^2 + K}{u^0} \right) + \frac{1}{c} \frac{\partial(\rho_{0q} \mathbf{A})}{\partial \mathbf{v}} \cdot \mathbf{v} \right] u^0 \sqrt{-g}\, dx^1 dx^2 dx^3 + \sum_{n=1}^{N} \frac{\partial L_f}{\partial \mathbf{v}_n}. \quad (22)$$

To obtain the equation of particles' motion, it is necessary to calculate the action variation in (9), which contains only variations of the four-currents, and to equate this variation to zero. Consequently, in (9) it is necessary to use only the first integral on the right side

$$\delta S_1 = \int_{t_1}^{t_2} \int_V \delta \mathcal{L} \sqrt{-g}\, dx^1 dx^2 dx^3 dt = 0. \quad (23)$$

The variation of Lagrangian density $\mathcal{L} = \mathcal{L}_p + \mathcal{L}_f$ with respect to four-currents reduces to the variation of $\mathcal{L}_p$, since $\mathcal{L}_f$ (2) does not depend on the four-currents. Taking into account (1) for variation $\mathcal{L}_p$ over four currents we find

$$\delta \mathcal{L}_p = -A_\mu \delta j^\mu - \frac{c\left(\delta J^\mu J^\nu g_{\mu\nu} + J^\mu \delta J^\nu g_{\mu\nu}\right)}{2\sqrt{J^\mu J^\nu g_{\mu\nu}}} - \frac{\partial K}{\partial J^\mu} \delta J^\mu = -A_\mu \delta j^\mu - u_\mu \delta J^\mu - \frac{\partial K}{\partial J^\mu} \delta J^\mu. \quad (24)$$

According to [36], the variations of four-currents are equal to

$$\delta j^\mu = \nabla_\sigma \left( j^\sigma \delta x^\mu - j^\mu \delta x^\sigma \right) = \frac{1}{\sqrt{-g}} \partial_\sigma \left[ \sqrt{-g} \left( j^\sigma \delta x^\mu - j^\mu \delta x^\sigma \right) \right],$$

$$\delta J^\mu = \nabla_\sigma \left( J^\sigma \delta x^\mu - J^\mu \delta x^\sigma \right) = \frac{1}{\sqrt{-g}} \partial_\sigma \left[ \sqrt{-g} \left( J^\sigma \delta x^\mu - J^\mu \delta x^\sigma \right) \right]. \quad (25)$$

Let's substitute into (24) $\delta j^\mu$ and $\delta J^\mu$ from (25), and then substitute $\delta \mathcal{L}_p$ into (23) instead of $\delta \mathcal{L}$

$$\delta S_1 = -\int_{t_1}^{t_2}\int_V \left\{ \begin{array}{l} A_\mu \frac{1}{\sqrt{-g}}\partial_\sigma \\ \left[ \begin{array}{l} \sqrt{-g} \\ \left( j^\sigma \delta x^\mu - j^\mu \delta x^\sigma \right) \end{array} \right] + \\ +\left( u_\mu + \frac{\partial K}{\partial J^\mu} \right)\nabla_\sigma \\ \left( J^\sigma \delta x^\mu - J^\mu \delta x^\sigma \right) \end{array} \right\} dt = 0. \quad (26)$$

$$\sqrt{-g}\, dx^1 dx^2 dx^3$$

Acting as in [36], a variation of the action and the equation of motion were found in Appendix A in (A11)

$$\delta S_1 = \int_{t_1}^{t_2}\int_V \left[ \begin{array}{l} j^\sigma F_{\sigma\mu} + J^\sigma \nabla_\sigma \left( u_\mu + \frac{\partial K}{\partial J^\mu} \right) \\ -J^\sigma \nabla_\mu \frac{\partial K}{\partial J^\sigma} \end{array} \right] dt = 0 .$$

$$\delta x^\mu \sqrt{-g}\, dx^1 dx^2 dx^3$$

$$J^\sigma \nabla_\sigma \left( u_\mu + \frac{\partial K}{\partial J^\mu} \right) = J^\sigma \nabla_\mu \frac{\partial K}{\partial J^\sigma} - j^\sigma F_{\sigma\mu} . \quad (27)$$

In (27), taking into account expression for the four-current $J^\sigma = \rho_0 u^\sigma$, we write the equation of motion in terms of operator of proper-time-derivative $u^\sigma \nabla_\sigma = \frac{D}{D\tau}$

$$\rho_0 \frac{D}{D\tau}\left( u_\mu + \frac{\partial K}{\partial J^\mu} \right) = J^\sigma \nabla_\mu \frac{\partial K}{\partial J^\sigma} - j^\sigma F_{\sigma\mu} . \quad (28)$$

In general relativity the stress-energy tensor $T_{\mu\nu} = \tau_{\mu\nu} + W_{\mu\nu}$ of matter with pressure $\mathbb{P}_0$ is used. In this case, the equation of motion is found by taking the divergence of this tensor: $\nabla^\nu T_{\mu\nu} = 0$. Considering the expressions for $W_{\mu\nu}$ (15) and for $\tau_{\mu\nu}$ (16), using the known equation for electromagnetic field $\nabla^\nu W_{\mu\nu} = j^\sigma F_{\sigma\mu}$, we express the equation of motion $\nabla^\nu T_{\mu\nu} = 0$ in terms of the pressure

$$\nabla^\nu \left[ \left( \rho_0 + \frac{\mathbb{P}_0}{c^2} \right) u_\mu u_\nu \right] = \nabla_\mu \mathbb{P}_0 - j^\sigma F_{\sigma\mu} . \quad (29)$$

Equation (29) is the equation of motion in general relativity for charged matter in scalar pressure field $\mathbb{P}_0$, in electromagnetic field with the tensor $F_{\sigma\mu}$ and in gravitational field, defined in terms of the metric tensor.

In derivation of equation of motion (27), we used the variations $\delta j^\mu$ and $\delta J^\mu$ from (25). According to [36], the expressions for these variations are valid on the condition that the continuity equations hold for the four-currents: $\nabla_\mu J^\mu = 0$, $\nabla_\mu j^\mu = 0$. Using the continuity equation $\nabla_\mu J^\mu = 0$, we substitute $j^\sigma F_{\sigma\mu}$ from ((29) into (27). This leads to the following

$$\begin{array}{l} J^\sigma \nabla_\sigma \frac{\partial K}{\partial J^\mu} - J^\sigma \nabla_\mu \frac{\partial K}{\partial J^\sigma} \\ = \frac{1}{c^2}\nabla^\nu \left( \mathbb{P}_0 u_\mu u_\nu \right) - \nabla_\mu \mathbb{P}_0 \end{array} . \quad (30)$$

If we multiply (30) by $u^\mu$, the left side in (30), taking into account the equalities $J^\sigma = \rho_0 u^\sigma$, $J^\mu = \rho_0 u^\mu$, vanishes

$$\begin{array}{l} u^\mu J^\sigma \nabla_\sigma \frac{\partial K}{\partial J^\mu} - u^\mu J^\sigma \nabla_\mu \frac{\partial K}{\partial J^\sigma} \\ = u^\mu J^\sigma \nabla_\sigma \frac{\partial K}{\partial J^\mu} - u^\sigma J^\mu \nabla_\sigma \frac{\partial K}{\partial J^\mu} = 0 \end{array} . \quad (31)$$

For the right side of (30) after multiplication by $u^\mu$ taking into account the equalities $u^\mu \nabla^\nu u_\mu = 0$, $u_\mu u^\mu = c^2$, the following is obtained

$$\begin{array}{l} \frac{1}{c^2} u^\mu \nabla^\nu \left( \mathbb{P}_0 u_\mu u_\nu \right) - u^\mu \nabla_\mu \mathbb{P}_0 \\ = \frac{1}{c^2} u^\mu u_\mu u_\nu \nabla^\nu \mathbb{P}_0 + \\ \frac{1}{c^2} \mathbb{P}_0 u^\mu \nabla^\nu \left( u_\mu u_\nu \right) - u^\mu \nabla_\mu \mathbb{P}_0 = \\ = u_\nu \nabla^\nu \mathbb{P}_0 + \frac{\mathbb{P}_0}{c^2} u^\mu \nabla^\nu \left( u_\mu u_\nu \right) \\ - u^\mu \nabla_\mu \mathbb{P}_0 = \\ \frac{\mathbb{P}_0}{c^2} u^\mu \nabla^\nu \left( u_\mu u_\nu \right) = \mathbb{P}_0 \nabla^\nu u_\nu . \end{array} \quad (32)$$

From (30-32) it follows that the equations of motion (27) and (29) are consistent provided that in (32) $\nabla^\nu u_\nu = 0$. Taking into account the continuity equation $\nabla^\nu \left( \rho_0 u_\nu \right) = 0$, we can see that in the system under consideration the condition $u_\nu \nabla^\nu \rho_0 = \frac{d\rho_0}{d\tau} = 0$ must be satisfied, that is, the mass density $\rho_0$ must be constant in the comoving reference frame of each matter element. This means that $\rho_0$ should not depend on the time and coordinates within each matter element.

These restrictions on the consistency of equations of motion (27) and (29) show that equation (29) in the general case does not represent a full description of motion of real matter. The same applies to the stress-energy tensor $\tau_{\mu\nu}$ (16), which in this case must contain additional specifying terms.

Equation (30) can be considered an equation for determining the value of $K$. Thus, provided that

$\nabla_\mu \rho_0 = 0$, the particular solution of (30) is $K = \frac{\mathbb{P}_0 \sqrt{J^\mu J^\nu g_{\mu\nu}}}{\rho_0 c} = \mathbb{P}_0$. To prove this, we must take into account $K = \frac{\mathbb{P}_0 \sqrt{J^\mu J^\nu g_{\mu\nu}}}{\rho_0 c}$, the condition $\nabla_\mu \rho_0 = 0$ and the following relations in (30)

$$\frac{\partial K}{\partial J^\mu} = \frac{2\mathbb{P}_0 J^\nu g_{\mu\nu}}{2\rho_0 c \sqrt{J^\mu J^\nu g_{\mu\nu}}} = \frac{\mathbb{P}_0 u_\mu}{\rho_0 c^2},$$

$$u^\mu \nabla^\nu u_\mu = 0, \qquad \nabla_\sigma J^\sigma = 0. \tag{33}$$

The condition $\nabla_\mu \rho_0 = 0$ is equivalent to the condition $\partial_\mu \rho_0 = \frac{\partial \rho_0}{\partial x^\mu} = 0$, which corresponds to matter with uniform density, for example, an incompressible liquid or a body densely composed of identical solid particles of constant mass density.

For free matter without an electromagnetic field and without considering the pressure, it follows from (29)

$$\nabla^\nu \left( \rho_0 u_\mu u_\nu \right) = u_\mu \nabla^\nu J_\nu + J_\nu \nabla^\nu u_\mu = \rho_0 \frac{D u_\mu}{D\tau} = 0 .$$

$$a_\mu = \frac{D u_\mu}{D\tau} = u^\sigma \nabla_\sigma u_\mu = u^\sigma \partial_\sigma u_\mu - \Gamma^\lambda_{\sigma\mu} u_\lambda u^\sigma = \frac{d u_\mu}{d\tau} - \Gamma^\lambda_{\sigma\mu} u_\lambda u^\sigma = 0 .$$

$$a^\mu = \frac{D u^\mu}{D\tau} = u^\sigma \nabla_\sigma u^\mu = u^\sigma \partial_\sigma u^\mu + \Gamma^\mu_{\sigma\lambda} u^\lambda u^\sigma = \frac{d u^\mu}{d\tau} + \Gamma^\mu_{\sigma\lambda} u^\lambda u^\sigma = 0 . \tag{34}$$

Equations (34) for the covariant four-acceleration $a_\mu$ and the contravariant acceleration $a^\mu$ show that the free matter, in the absence of external fields and without taking into account the internal pressure, moves with zero four-acceleration along the so-called geodesic line. This means that the gravitational field changes synchronously with changing metric tensor in such a way that the small test particles move in the same way regardless of their mass, when all other conditions being equal. However, in the presence of external nongravitational fields, taking into account the pressure and sufficiently large test particles, the equations (34) will no longer hold true.

The latter follows from the fact that the metric inside a test particle arises not only from the action of external gravitational field, in which the particle is moving but also from the particle's own gravitational field. Gravitation inevitably changes the internal pressure $\mathbb{P}_0$ in matter, the pressure gradients create internal forces, and the four-acceleration becomes nonzero. Equations (34) are equations of motion for a single point particle, but not for real matter, for which (27) should be used together with (30) to determine the relationship of the function $K(J^\mu, g_{\mu\nu})$ with pressure $\mathbb{P}_0$.

The presence of an electromagnetic field manifests itself in general relativity in two ways – on the one hand, the metric tensor and corresponding gravitational field change; on the other hand, the charged particles experience the Lorentz force and generate electromagnetic radiation. Thus, in the general case, the motions of neutral and charged particles differ significantly from each other.

The results obtained above will not change, if we use as $\mathcal{L}_p$ the following expression

$$\mathcal{L}_p = -A_\mu j^\mu - c\rho_0 \sqrt{u^\mu u^\nu g_{\mu\nu}} - K(J^\mu, g_{\mu\nu}) = -\frac{1}{c}\rho_{0q} u^0 \varphi + \frac{1}{c}\rho_{0q} u^0 \mathbf{A}\cdot\mathbf{v} - c\rho_0 \sqrt{u^\mu u^\nu g_{\mu\nu}} - K(J^\mu, g_{\mu\nu}). \tag{35}$$

In (35) the quantity $c\rho_0 \sqrt{u^\mu u^\nu g_{\mu\nu}}$ is used instead of $c\sqrt{J^\mu J^\nu g_{\mu\nu}}$ in (1); moreover, the result of variation in the action remains the same.

## 3. Results

### 3.1. GTR[1] version

In this Section we consider the GTR[1] version, which is the closest version to standard general relativity. In the analysis of GTR[1] we rely on the results of Section 2, obtained from the principle of least action.

The equation for the metric in general relativity, which contains the cosmological constant $\Lambda_{GR}$, has the following form [36-39]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda_{GR} g_{\mu\nu} = -\frac{1}{2ck} T_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} = \varkappa\left( \tau_{\mu\nu} + W_{\mu\nu} \right), \tag{36}$$

where the tensor $T_{\mu\nu} = \tau_{\mu\nu} + W_{\mu\nu}$ is the sum of stress-energy tensor (16) of matter and stress-energy tensor (15) of electromagnetic field, $k = -\frac{c^3}{16\pi G} = -\frac{1}{2c\varkappa}$.

It should be noted that the left side of (36) consists of geometric quantities associated with the spacetime metric, and physical quantities are concentrated in the right side of (36). The coefficient $\varkappa = \frac{8\pi G}{c^4}$ in front of stress-energy tensor $T_{\mu\nu}$ in (36) was chosen so that the general relativity in the weak field limit reproduces Newton's law of gravitation. However, this eliminates in advance those small additives that may be present in the value of coefficient $\varkappa$. This means that in fact the coefficient $\varkappa$ should be considered an unknown quantity, which should be derived from the general relativity itself and from experiment, without relying on a less accurate Newton theory. In this regard, in the theory of vector fields, which will be discussed in Section 3.4, it is assumed that $\varkappa = \frac{8\pi G\beta}{c^4}$, where $\beta$ is a constant coefficient to be determined.

Let us equate the identical terms in (36) and (17) and multiply the result by $g^{\mu\nu}$. Hence it follows that both equations coincide under the following condition

$$8c\,k\,\Lambda_{GR} = 4A_\beta\, j^\beta + 4\rho_0 c^2 + 3\mathbb{P}_0 + 4K + 2\frac{\partial K}{\partial g_{\alpha\beta}} g_{\alpha\beta} + 8c\,k\,\Lambda . \qquad (37)$$

We apply contraction of equation (36) by multiplying by $g^{\mu\nu}$ and by taking into account the tensor expressions (15-16)

$$2c\,kR = 8c\,k\Lambda_{GR} + \rho_0 c^2 - 3\mathbb{P}_0 . \qquad (38)$$

Substitution of $8c\,k\Lambda_{GR}$ from (37) into ((38) allows us to express the scalar curvature $R$ inside matter of the physical system in terms of $\Lambda$:

$$2c\,kR = 5\rho_0 c^2 + 4A_\beta\, j^\beta + 4K + 2\frac{\partial K}{\partial g_{\alpha\beta}} g_{\alpha\beta} + 8c\,k\,\Lambda . \qquad (39)$$

Equation (39) coincides with (18). We substitute $\Lambda_{GR}$ from (38) into (36):

$$R_{\mu\nu} - \frac{1}{4}R\,g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} - \frac{2\pi G}{c^4}\left(\rho_0 c^2 - 3\mathbb{P}_0\right) g_{\mu\nu} . \qquad (40)$$

By solving equation (40), we can calculate the metric tensor inside matter and determine scalar curvature $R$. Next, with the known value $R$, we find $\Lambda_{GR}$ from (38). If equation (30) is solved and the function $K$ is determined, then from (39) we obtain the expression for $\Lambda$.

Let us consider the situation outside matter. In this case, according to (37), the following equality holds true: $\Lambda_{GR} = \Lambda$. Then, the equality $R = 4\Lambda_{GR} = 4\Lambda$ follows from (38). The equation for the metric (36), as well as equation (17), take the following form:

$$R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} = \frac{8\pi G}{c^4}W_{\mu\nu} . \qquad (41)$$

In (41) the stress-energy tensor $W_{\mu\nu}$ of electromagnetic field changes the spacetime curvature outside matter. The solution of equations (40-41) is a dependence of the metric tensor on coordinates and time, while at the points on the surface surrounding matter, the metric tensor components in both equations, due to their equality, must coincide with each other. This allows us to determine part of the unknown constants in solutions for the metric tensor inside and outside matter.

We now turn to the formula for the system's energy (20) and we substitute $\Lambda$ with $R$ with the help of (39). Considering that $A_\mu\, j^\mu = \frac{1}{c}\rho_{0q}u^0\varphi - \frac{1}{c}\rho_{0q}u^0 \mathbf{A}\cdot\mathbf{v}$, we find the following for the energy inside matter

$$E_i = \int_V \left[ \frac{1}{c}\rho_{0q}\mathbf{A} - \frac{\partial}{\partial \mathbf{v}}\left( \frac{1}{c}\rho_{0q}\varphi + \frac{\rho_0 c^2 + K}{u^0} \right) + \frac{1}{c}\frac{\partial\left(\rho_{0q}\mathbf{A}\right)}{\partial \mathbf{v}}\cdot\mathbf{v} \right] \cdot \mathbf{v}\, u^0 \sqrt{-g}\, dx^1 dx^2 dx^3$$
$$+ \int_V \left( -\frac{\rho_0 c^2}{4} - \frac{1}{2}\frac{\partial K}{\partial g_{\alpha\beta}} g_{\alpha\beta} + \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}c\,kR \right) \sqrt{-g}\, dx^1 dx^2 dx^3 + \sum_{n=1}^{N}\left( \mathbf{v}_n \cdot \frac{\partial L_f}{\partial \mathbf{v}_n} \right). \qquad (42)$$

Outside matter, according to (39), $R = 4\Lambda$, so that the energy (20) is written as follows:

$$E_o = \int_V \left( \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}c\,k\,R \right) \sqrt{-g}\, dx^1 dx^2 dx^3 + \sum_{n=1}^{N}\left( \mathbf{v}_n \cdot \frac{\partial L_f}{\partial \mathbf{v}_n} \right) . \qquad (43)$$

To determine the system's energy with the help of (42-43), we need to know the dependence of the scalar curvature $R$ on the coordinates and time; that is, first, we need to solve equations for the metrics (40-41). When calculating the total relativistic energy of a physical system, it is necessary to sum up the energies (42) and (43): $E = E_i + E_o$. Since (43) can be obtained from (42) at $\rho_0 = 0$, $K = 0$ and $\rho_{0q} = 0$, that is, in the absence of matter, (42) is the general formula for the energy in standard general relativity.

The presence of the scalar curvature $R$ in the formulas for energy (42-43) is necessary to take into account contribution of gravitational energy to the total system's energy with the help of metric.

The relativistic momentum $\mathbf{P}$ of a system, according to (22), depends on the scalar curvature $R$ and the cosmological constant $\Lambda$ only through the sum $\sum_{n=1}^{N} \frac{\partial L_f}{\partial \mathbf{v}_n}$. After calculating the energy $E$ and the momentum $\mathbf{P}$, it is possible to determine the system's four-momentum of the system, defined in [8] as $P_\mu = \left( \frac{E}{c}, -\mathbf{P} \right)$. The equation for determining the motion of matter in GTR[1] is equation (29).

### 3.2. GTR[2] version

In this Section we proceed not from equation (36) for the metric of standard general relativity, but rather from the derivation of general relativity with the help of Lagrangian formalism in Section 2. This will lead us to a new version of general relativity, which we denote GTR[2]. Let us express $\Lambda$ from (18) and $\tau_{\mu\nu}$ from (16), and substitute them into (17)

$$2c\,k\,R_{\mu\nu} - c\,k\,R\,g_{\mu\nu} = -g_{\mu\nu}\left( \frac{1}{2}c\,k\,R - \frac{1}{4}\rho_0 c^2 - \frac{1}{2}\frac{\partial K}{\partial g_{\alpha\beta}} g_{\alpha\beta} \right) - 2\frac{\partial K}{\partial g_{\alpha\beta}} g_{\mu\alpha} g_{\nu\beta} - \rho_0 u_\mu u_\nu - W_{\mu\nu}. \quad (44)$$

Transposing the term containing $R$ from the right-hand side (44) to the left-hand side, taking into account the coefficient $k = -\frac{c^3}{16\pi G}$, we obtain equation for the metric

$$R_{\mu\nu} - \frac{1}{4}R\,g_{\mu\nu} = -\frac{2\pi G}{c^4}\left( \rho_0 c^2 + 2\frac{\partial K}{\partial g_{\alpha\beta}} g_{\alpha\beta} \right) g_{\mu\nu} + \frac{8\pi G}{c^4}\left( 2\frac{\partial K}{\partial g_{\alpha\beta}} g_{\mu\alpha} g_{\nu\beta} + \rho_0 u_\mu u_\nu + W_{\mu\nu} \right). \quad (45)$$

The left-hand side of (44) contains the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$ multiplied by $2ck$. The well-known property of this tensor is that its divergence is equal to zero: $\nabla^\nu G_{\mu\nu} = 0$. Consequently, the divergence of the right-hand side of (44) must also be equal to zero

$$\nabla_\mu \left( \frac{1}{2}c\,k\,R - \frac{1}{4}\rho_0 c^2 - \frac{1}{2}\frac{\partial K}{\partial g_{\alpha\beta}} g_{\alpha\beta} \right) + 2g_{\mu\alpha}\nabla_\beta \frac{\partial K}{\partial g_{\alpha\beta}} + \nabla^\nu \left( \rho_0 u_\mu u_\nu \right) + \nabla^\nu W_{\mu\nu} = 0. \quad (46)$$

In (46) we used expression $g_{\mu\nu}\nabla^\nu = \nabla_\mu$ and the fact that under covariant differentiation the metric tensor behaves as a constant. We substitute the equation of motion (27) into (46) and take into account the equality for electromagnetic field $\nabla^\nu W_{\mu\nu} = j^\sigma F_{\sigma\mu}$, the continuity equation $\nabla^\nu \left( \rho_0 u_\nu \right) = 0$ and the equality $2\rho_0 u_\nu \nabla^\nu u_\mu = 2J^\sigma \nabla_\sigma u_\mu$

$$c\,k\,\nabla_\mu R = \nabla_\mu \left( \frac{1}{2}\rho_0 c^2 + \frac{\partial K}{\partial g_{\alpha\beta}} g_{\alpha\beta} \right) - 4g_{\mu\alpha}\nabla_\beta \frac{\partial K}{\partial g_{\alpha\beta}} + 2J^\sigma \nabla_\sigma \frac{\partial K}{\partial J^\mu} - 2J^\sigma \nabla_\mu \frac{\partial K}{\partial J^\sigma}. \quad (47)$$

Now we take the covariant derivative $\nabla^\nu$ of both sides of (17) and take into account the fact that for the cosmological constant it must be $\nabla^\nu \left( \Lambda g_{\mu\nu} \right) = \nabla_\mu \Lambda = 0$. Using in the right-hand side of (17) $\tau_{\mu\nu}$ from (16) and equality $\nabla^\nu W_{\mu\nu} = j^\sigma F_{\sigma\mu}$, we find

$$\nabla_\mu \left( A_\beta\, j^\beta + \rho_0 c^2 + K \right) + 2g_{\mu\alpha}\nabla_\beta \left( \frac{\partial K}{\partial g_{\alpha\beta}} \right) + \nabla^\nu \left( \rho_0 u_\mu u_\nu \right) + j^\sigma F_{\sigma\mu} = 0. \quad (48)$$

A comparison of (48) with equation of motion (27) gives

$$\nabla_\mu \left( A_\beta\, j^\beta + \rho_0 c^2 + K \right) + 2g_{\mu\alpha}\nabla_\beta \frac{\partial K}{\partial g_{\alpha\beta}} = J^\sigma \nabla_\sigma \frac{\partial K}{\partial J^\mu} - J^\sigma \nabla_\mu \frac{\partial K}{\partial J^\sigma}. \quad (49)$$

Multiplication (49) by the four-velocity $u^\mu$ leads to the following

$$u^{\mu}\nabla_{\mu}\left(A_{\beta}\, j^{\beta}+\rho_0 c^2+K\right)+2u_{\alpha}\nabla_{\beta}\frac{\partial K}{\partial g_{\alpha\beta}}=0 . \qquad (50)$$

We can assume that (45), (47) and (49-50) are the system of equations that allows us to simultaneously find the function $K$ and the metric tensor $g_{\alpha\beta}$. After $K$ and $g_{\alpha\beta}$ are found, we can use them in the equation of motion (27).

If we express $\Lambda$ from (18) and substitute it into (20), we obtain an expression coinciding with the energy (42) in the GTR[1] version. Similarly, the expression for the momentum (22) is remain unchanged.

Outside matter, it follows from (18) that $R=4\Lambda$ , and the equation for metric (44) becomes the same as that in (41) in the GTR[1] version. In this case, according to (47), $\nabla_{\mu}R=\partial_{\mu}R=0$, and we can assume that the scalar curvature $R$ is a constant. However, inside matter, the scalar curvature is a scalar function of the coordinates and time.

### 3.3. Discussion of GTR[1] and GTR[2] versions

Let's first consider the GTR[1] version. By definition, the cosmological constant $\Lambda$ does not depend on time or coordinates, which are taken into account during variation in the principle of least action. It follows from (37) that the quantity $\Lambda_{GR}$ is equal to the cosmological constant $\Lambda$ only outside matter. However, inside matter, $\Lambda_{GR}$ is no longer constant and becomes a certain scalar function, depending on the coordinates and time, such that $\nabla_{\mu}\Lambda_{GR}\neq 0$. The latter also applies to the scalar curvature $R$ , according to (38).

We rewrite the equation for the metric (40) in terms of the Einstein tensor $G^{\mu\nu}=R^{\mu\nu}-\frac{1}{2}Rg^{\mu\nu}$ , for which purpose we subtract the quantity $\frac{1}{4}Rg_{\mu\nu}$ from both parts of (40), taking into account the coefficient $2ck=-\frac{c^4}{8\pi G}$

$$2c\,kG_{\mu\nu}=-\left(\frac{1}{2}c\,kR-\frac{1}{4}\rho_0 c^2+\frac{3}{4}\mathbb{P}_0\right)g_{\mu\nu}-T_{\mu\nu} . \qquad (51)$$

Let us take the covariant derivative $\nabla^{\nu}$ of both sides of (51). The covariant derivative of the left-hand side will be equal to zero, since $\nabla^{\nu}G_{\mu\nu}=0$ due to the property of the Einstein tensor. On the other hand, the covariant derivative of stress-energy tensor is equal to zero, $\nabla^{\nu}T_{\mu\nu}=0$, which leads to equation of motion (29). Taking into account the additivity of covariant derivative with respect to the sum of tensors, for remaining terms on the right-hand side (51) we have:

$$\nabla_{\mu}\left(2c\,kR-\rho_0 c^2+3\mathbb{P}_0\right)=0 . \qquad (52)$$

The equation (52) represents an additional limitation on the value of the scalar curvature $R$ inside matter in GTR[1] version and relates $R$ to the matter’s density and pressure.

In standard general relativity it is assumed that the equation for metric (36) is the primary equation. However, we agree with the validity of (36) only in the case of uncharged dust-like matter in the absence of pressure between the particles and at constant mass density $\rho_0$, when the stress-energy tensor of matter has the form $\tau'_{\mu\nu}=\rho_0\, u_{\mu}u_{\nu}$. In this case, according to (37), $2ck\Lambda_{GR}=\rho_0 c^2+2ck\Lambda$, and the equation of motion $\nabla^{\nu}\tau'_{\mu\nu}=0$ follows from the vanishing of divergence of the right-hand side of (36); this equation coincides with the equation of motion (34) of the free matter.

In the case of continuously distributed matter with scalar pressure, we could not find a Lagrangian density that would give, as a result of applying the principle of least action, equation (36) for the metric in general relativity, while fulfilling two conditions: 1) The stress-energy tensor $\tau_{\mu\nu}=\left(\rho_0+\frac{\mathbb{P}_0}{c^2}\right)u_{\mu}u_{\nu}-\mathbb{P}_0\, g_{\mu\nu}$ must be present in (36); 2) $\Lambda_{GR}$ must not be a scalar function, but rather a constant value and a real cosmological constant. The analysis of the literature gives the impression that the stress-energy tensor $\tau_{\mu\nu}=\left(\rho_0+\frac{\mathbb{P}_0}{c^2}\right)u_{\mu}u_{\nu}-\mathbb{P}_0\, g_{\mu\nu}$ is inserted into the equation for the metric (36) “manually”, simply by analogy with the case of uncharged dust-like matter, without accurate derivation from the principle of least action.

To be more precise, we will remember that in hydrodynamics, the following Lagrangian density is sometimes used

$$\mathcal{L}_m=-\rho_0 c^2-\rho_0\int_0^{\mathbb{P}_0}\frac{d\,\mathbb{P}_0}{\rho_0}+\mathbb{P}_0=-\rho_0 c^2-\rho_0\Pi . \qquad (53)$$

In [36], the quantity $\Pi=\int_0^{\mathbb{P}_0}\frac{d\mathbb{P}_0}{\rho_0}-\frac{\mathbb{P}_0}{\rho_0}$ is the potential energy of a fluid elastic compression, which refers to the mass unit, while $\mathbb{P}_0=\rho_0^2\frac{d\Pi}{d\rho_0}$. In [40] an isentropic perfect fluid was considered, and instead of $\Pi$, a similar quantity $\varepsilon=\Pi/c^2$ was used. Variation in the

Lagrangian density $\mathcal{L}_m$ (53) gives the stress-energy tensor

$$\tau_{\mu\nu} = \left( \rho_0 + \frac{\rho_0}{c^2} \int_0^{\mathbb{P}_0} \frac{d\mathbb{P}_0}{\rho_0} \right) u_\mu u_\nu - \mathbb{P}_0 g_{\mu\nu} \qquad (54)$$

and the equation of motion

$$\left( \rho_0 + \frac{\rho_0}{c^2} \int_0^{\mathbb{P}_0} \frac{d\mathbb{P}_0}{\rho_0} \right) u_\nu \nabla^\nu u_\mu = \nabla_\mu \mathbb{P}_0 - \frac{u_\mu u_\nu}{c^2} \nabla^\nu \mathbb{P}_0 . \qquad (55)$$

If in (54-55) we assume that the mass density $\rho_0$ does not depend on the pressure $\mathbb{P}_0$, take into account the continuity equation $\nabla^\nu (\rho_0 u_\nu) = \nabla^\nu J_\nu = 0$ and condition $\nabla^\nu u_\nu = 0$, then (54-55) would coincide, respectively, with the stress-energy tensor (16) and equation of motion (29), taken without regard to electromagnetic field. Thus, for the Lagrangian density $\mathcal{L}_m$ (53) to actually lead to the stress-energy tensor and the equation of motion required in general relativity, it is necessary to satisfy the condition of constant mass density in the form $\nabla^\nu \rho_0 = 0$. In the general case, when $\nabla^\nu \rho_0 \neq 0$, the Lagrangian density $\mathcal{L}_m$ cannot be the Lagrangian density of general relativity.

To understand this problem, we constructed Lagrangian density $\mathcal{L} = \mathcal{L}_p + \mathcal{L}_f$ (1-2) and introduced the function $K(J^\mu, g_{\mu\nu})$, which leads to the emergence of pressure force in matter and is present in the equation of motion. Now, suppose that the cosmological constant $\Lambda_{GR}$ is still a constant value in general relativity. Then, (37) is an equation of the state of matter, since it relates, with an accuracy of up to a constant, the mass density, the pressure, and the energy density of electromagnetic current.

The equation of motion (29), which is a consequence of equation $\nabla^\nu T_{\mu\nu} = 0$, is consistent with equation of motion (27), which is derived from the principle of least action, only when $\nabla^\nu u_\nu = 0$, $\frac{d\rho_0}{d\tau} = 0$, which is equivalent to the relation $\frac{d\rho_0}{dt} = \frac{\partial \rho_0}{\partial t} + \mathbf{v} \cdot \nabla \rho_0 = 0$.

In this case the function $K(J^\mu, g_{\mu\nu})$ becomes equal to the pressure $\mathbb{P}_0$: $K = \frac{\mathbb{P}_0 \sqrt{J^\mu J^\nu g_{\mu\nu}}}{\rho_0 c} = \mathbb{P}_0$. With this in mind, we take the derivative $\frac{\partial K}{\partial g_{\mu\nu}}$ and substitute it into (37):

$$\frac{\partial K}{\partial g_{\mu\nu}} = \frac{\mathbb{P}_0 J^\mu J^\nu}{2\rho_0 c \sqrt{J^\mu J^\nu g_{\mu\nu}}} = \frac{\mathbb{P}_0}{2c^2} u^\mu u^\nu ,$$

$$\rho_0 c^2 + A_\beta j^\beta + \mathbb{P}_0 = 2c\,k\,\Lambda_{GR} - 2c\,k\,\Lambda = const . \qquad (56)$$

The last equality in (56), as an equation of state of matter, cannot be considered the general expression, which limits the applicability of general relativity approach with its equation for metric (36), stress-energy tensor of matter (16), and equation of motion (29).

As a result, we are faced with a number of paradoxical conclusions about general relativity, the validity of which appears to be questionable and which we suggest taking on faith. For example, let us assume that in (36) $\Lambda_{GR}$ is a constant value and that $\tau_{\mu\nu} = \left( \rho_0 + \frac{\mathbb{P}_0}{c^2} \right) u_\mu u_\nu - \mathbb{P}_0 g_{\mu\nu}$, as is assumed in general relativity in (16). Then, the divergence of the left-hand side of (36) is zero, and the equality to zero of divergence of the right-hand side (36) in form $\nabla^\nu T_{\mu\nu} = 0$ gives us equation (29), which can be written as follows

$$\left( \rho_0 + \frac{\mathbb{P}_0}{c^2} \right) u_\nu \nabla^\nu u_\mu + u_\mu \nabla^\nu \left[ \left( \rho_0 + \frac{\mathbb{P}_0}{c^2} \right) u_\nu \right] - \nabla_\mu \mathbb{P}_0 + j^\sigma F_{\sigma\mu} = 0 . \qquad (57)$$

By multiplying (57) by the four-velocity $u^\mu$ and taking into account that $j^\sigma = \rho_{0q} u^\sigma$, $u^\mu \nabla^\nu u_\mu = 0$, $u^\mu u_\mu = c^2$, and

$$j^\sigma F_{\sigma\mu} u^\mu = \rho_{0q} u^\mu F_{\mu\sigma} u^\sigma = -\rho_{0q} u^\mu F_{\sigma\mu} u^\sigma = -\rho_{0q} u^\sigma F_{\sigma\mu} u^\mu = 0$$

as a consequence of the antisymmetry of tensor $F_{\sigma\mu}$, then we obtain the following result:

$$\nabla^\nu \left[ \left( \rho_0 c^2 + \mathbb{P}_0 \right) u_\nu \right] - u^\mu \nabla_\mu \mathbb{P}_0 = 0 . \qquad (58)$$

The equation (58) can be simplified by permutation and substitution of the indices: $\nabla^\nu \left[ \left( \rho_0 c^2 + \mathbb{P}_0 \right) u_\nu \right] = \nabla_\mu \left[ \left( \rho_0 c^2 + \mathbb{P}_0 \right) u^\mu \right]$. This gives us the following:

$$\nabla_\mu \left( \rho_0 u^\mu \right) + \frac{\mathbb{P}_0}{c^2} \nabla_\mu u^\mu = 0 . \qquad (59)$$

The expression (59) is considered in general relativity as a relativistic definition of continuity equation. However, (59) contradicts the continuity equation in the form $\nabla_\mu \left( \rho_0 u^\mu \right) = 0$, which was used for the action variation and finding equation of motion (27). In addition, as mentioned above, for the consistency of equations (27) and (29); the following conditions should be met: $\nabla_\mu u^\mu = 0$, $\nabla_\mu \rho_0 = 0$. If these conditions are not met, then equation of motion (29) of general relativity cannot be derived from the principle of least action; consequently, (29) becomes an assumed but unproved equation.

The condition $\nabla_\mu \rho_0 = 0$ corresponds to the condition of constancy of the mass density, which is possible, for example, in the relativistic uniform model. In this case, the models of compact stars using the general relativity will need correction if they are applied to matter with nonuniform density $\rho_0$.

Given that the standard general relativity is derived from the equation for metric (36), and not from the principle of least action, neither formula (19) for the energy $E$, nor formula (21) for the momentum $\mathbf{P}$ are used in general relativity.

Instead, a different approach is used in general relativity. It is assumed in [36-38] that the time components of stress-energy tensor $T_{\mu\nu} = \tau_{\mu\nu} + W_{\mu\nu}$ in (36) during integrating them over the four-volume can fully replace the system's four-momentum and give the energy $E$ and the momentum $\mathbf{P}$ for matter and nongravitational fields. As a consequence, the system mass is related to the volume integral of the energy density in the time component of stress-energy tensor of matter.

For the energy and momentum of gravitational field itself, the pseudotensor $t_{\mu\nu}$ is supposed to be used for calculation. A well-known problem of this approach is that the pseudotensor of the gravitational field is not a uniquely determined value. For example, in [9], seven different pseudotensors were referenced. It is pointed out that the problem of impossibility of unambiguous spatial localization of gravitational energy and the emergence of a pseudotensor instead of an energy-momentum tensor is due to the fact that gravitational field is "hidden" in the metric tensor.

In [41] it was emphasized that the gravitational field energy, found with the help of a pseudotensor under condition of constant matter density, is consistent with physical expectations, but differs if other equations of state of matter are used. In [42-43], it was proven that in general relativity, it is impossible to uniquely calculate the energy and mass of any arbitrarily chosen small part of the system. To the best of our knowledge, the questions of whether the system's energy and momentum, calculated in general relativity for continuously distributed matter taking into account the pressure and the pseudotensor of gravitational field, are truly equal to their values in formula (19) for energy $E$ and in formula (21) for momentum $\mathbf{P}$ have not yet been studied.

In cosmology, the equation (36) of general relativity for the metric is sometimes written as follows

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda_{GR}\, g_{\mu\nu}. \qquad (60)$$

Here, the cosmological constant $\Lambda_{GR}$ is used to describe dark energy, the nature of which is unknown but which modifies the equation for metric in accordance with observations. Let us substitute $\Lambda_{GR}$ (56) into (60)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} + \frac{8\pi G}{c^4} \left( 2\rho_0 c^2 + A_\beta\, j^\beta + \mathbb{P}_0 \right) g_{\mu\nu}. \qquad (61)$$

In (61) a value is added to the stress-energy tensor $T_{\mu\nu}$, which is proportional to rest energy density of cosmological matter $\rho_0 c^2$, the energy density of particles' four-current $A_\beta\, j^\beta$ and the pressure $\mathbb{P}_0$. This allows us to explain the meaning of mysterious dark energy – it appears on the right side of (61) in the form of a term $\left( 2\rho_0 c^2 + A_\beta\, j^\beta + \mathbb{P}_0 \right) g_{\mu\nu}$ as a consequence of the fact that the equation for metric (36) is actually not derived from the principle of least action, and $\Lambda_{GR}$ according to (37) turns out to be a scalar function and is not a real cosmological constant.

We now turn to characteristics of GTR[2]. In this version, equation for metric (45) and equation of motion (27) are used, derived from the principle of least action with the help of function $K(J^\mu, g_{\mu\nu})$. The GTR[2] version is more accurate and consistent than the GTR[1] version. One drawback of GTR[2] is the need to determine specific form of its function $K(J^\mu, g_{\mu\nu})$. The disadvantage of both versions of general relativity is that we first need to solve equation for the metric and to find the scalar curvature $R$ so that we can calculate the energy $E$ and momentum $\mathbf{P}$ of a system using formulas (19) and (21), respectively. This is a consequence of the fact that gravitational field is included in the metric tensor.

We can achieve even greater accuracy in GTR[2] if, instead of the function $K(J^\mu, g_{\mu\nu})$, which specifies the scalar pressure in Lagrangian density, we use corresponding terms for pressure as for a vector field, that is, use the four-potential and the pressure field tensor.

### 3.4. GTR$^m$ version

In this Section we consider modernized general theory of relativity, which we have designated GTR$^m$. Our goal will be to derive from the principle of least action equations of the theory for continuous matter, taking into account pressure and electromagnetic field in curved spacetime.

As mentioned in the previous Section, representation of the pressure as a scalar field has the disadvantage that it becomes necessary to determine the function $K(J^\mu, g_{\mu\nu})$ simultaneously with calculating spacetime metric tensor in a system of coupled equations. Moreover, the transition from the scalar pressure field to the vector pressure field increases accuracy of calculations and simplifies solution of equations. The same applies to acceleration field. Thus, our modernization of standard Lagrangian density of general relativity will consist of introducing the terms that turn the scalar fields into vector fields. This means that Lagrangian density will now include the four-potentials of acceleration field and pressure field, as well as corresponding tensor invariants of these fields. In this case, the Lagrangian density $\mathcal{L}' = \mathcal{L}'_p + \mathcal{L}'_f$ of GTR$^m$ version differs from the Lagrangian density (3) for vector fields only because of the absence of terms $D_\mu J^\mu + \frac{c^2}{16\pi G}\Phi_{\mu\nu}\Phi^{\mu\nu}$ for gravitational field

$$\mathcal{L}'_p = -A_\mu j^\mu - U_\mu J^\mu - \pi_\mu J^\mu = \frac{u^0}{c}\begin{pmatrix} -\rho_{0q}\varphi + \rho_{0q}\mathbf{A}\cdot\mathbf{v} - \\ \rho_0\vartheta + \rho_0\mathbf{U}\cdot\mathbf{v} - \rho_0\wp + \rho_0\mathbf{\Pi}\cdot\mathbf{v} \end{pmatrix}. \quad (62)$$

$$\mathcal{L}'_f = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - \frac{c^2}{16\pi\eta}u_{\mu\nu}u^{\mu\nu} - \frac{c^2}{16\pi\sigma}f_{\mu\nu}f^{\mu\nu} + ckR - 2ck\Lambda . \quad (63)$$

The main characteristic feature of general relativity is that the spacetime curvature plays the role of gravitational field, which is taken into account with the help of metric field given by the metric tensor and its derivatives with respect to $x^\mu$. While deriving GTR$^m$, we can almost fully use the results obtained for vector fields. Thus, the standard equations of electromagnetic field and similar equations for acceleration field and pressure field, presented in [1], remain in force. Moreover, after varying the Lagrangian density $\mathcal{L}' = \mathcal{L}'_p + \mathcal{L}'_f$ (62-63) with respect to the metric tensor in the principle of least action, the equation for metric is obtained in the following form:

$$2ckR_{\mu\nu} - ckRg_{\mu\nu} = -g_{\mu\nu}\begin{pmatrix} A_\alpha j^\alpha + U_\alpha J^\alpha + \\ \pi_\alpha J^\alpha + 2ck\Lambda \end{pmatrix} - T'_{\mu\nu}. \quad (64)$$

In (64) there is a total stress-energy tensor $T'_{\mu\nu} = W_{\mu\nu} + B_{\mu\nu} + P_{\mu\nu}$ that takes into account the stress-energy tensors of electromagnetic field $W_{\mu\nu}$, acceleration field $B_{\mu\nu}$ and pressure field $P_{\mu\nu}$. The only difference between (64) and equation for the metric for vector fields in [1] is that in (64) there is no contribution from the four-potential and from the stress-energy tensor of gravitational field. This is due to the fact that in the Lagrangian density (62-63) there are no terms that define gravitational field, except for scalar curvature $R$ and the metric tensor.

Contracting equation (64) with the metric tensor $g^{\mu\nu}$ gives the following

$$ckR = 2\begin{pmatrix} A_\alpha j^\alpha + U_\alpha J^\alpha \\ + \pi_\alpha J^\alpha + 2ck\Lambda \end{pmatrix}. \quad (65)$$

Equation (65) allows us to simplify the equation for metric (64). Considering the equality $2ck = -\frac{c^4}{8\pi G\beta}$, where $\beta$ is a coefficient of the order of unity, we find:

$$R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} = \frac{8\pi G\beta}{c^4}T'_{\mu\nu}. \quad (66)$$

It is not at first clear in general relativity if the cosmological constant $\Lambda$ can be used to gauge energy, as was done for vector fields in [1]. Let us turn to general expression for the energy of a system (19), into which we substitute the Lagrangian density (62-63). This for energy in GTR$^m$ version gives the following:

$$E' = \frac{1}{c}\int_V \begin{bmatrix} -\frac{\partial}{\partial\mathbf{v}}\left(\rho_{0q}\varphi + \rho_0\vartheta + \rho_0\wp\right) \\ +\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{v}}\left(\rho_{0q}\mathbf{A} + \rho_0\mathbf{U} + \rho_0\mathbf{\Pi}\right) \end{bmatrix} \cdot\mathbf{v}u^0\sqrt{-g}\,dx^1dx^2dx^3$$
$$+\int_V \begin{pmatrix} \frac{u^0}{c}\left(\rho_{0q}\varphi + \rho_0\vartheta + \rho_0\wp\right) \\ +\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} + \frac{c^2}{16\pi\eta}u_{\mu\nu}u^{\mu\nu} + \\ +\frac{c^2}{16\pi\sigma}f_{\mu\nu}f^{\mu\nu} - ckR + 2ck\Lambda \end{pmatrix} \sqrt{-g}\,dx^1dx^2dx^3$$
$$+\sum_{n=1}^{N}\left(\mathbf{v}_n\cdot\frac{\partial L'_f}{\partial\mathbf{v}_n}\right). \quad (67)$$

In (67), there is a quantity

$$L'_f = \int_V \mathcal{L}'_f\sqrt{-g}\,dx^1dx^2dx^3 = \int_V \begin{pmatrix} -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - \frac{c^2}{16\pi\eta}u_{\mu\nu}u^{\mu\nu} - \\ \frac{c^2}{16\pi\sigma}f_{\mu\nu}f^{\mu\nu} + ckR - 2ck\Lambda \end{pmatrix}, \sqrt{-g}\,dx^1dx^2dx^3 \quad (68)$$

which is that part of Lagrangian for which $\mathcal{L}'_f$ (63) is used.

The difference $2ck\Lambda - ckR$ in (67) must define the contribution of gravitational field to energy of the system, while the quantities $\Lambda$ and $R$ must satisfy (65); therefore, they can no longer be chosen arbitrarily for energy gauging. With the help of (65) we can exclude $\Lambda$ in (67). Taking into account relations of the type

$A_\alpha\ j^\alpha = \frac{1}{c}\rho_{0q}u^0\varphi - \frac{1}{c}\rho_{0q}u^0\mathbf{A}\cdot\mathbf{v}$ for all the fields, we find:

$$E_i' = \frac{1}{c}\int_V \begin{bmatrix} \rho_{0q}\mathbf{A} + \rho_0\mathbf{U} + \rho_0\mathbf{\Pi} - \\ \frac{\partial}{\partial \mathbf{v}}\left(\rho_{0q}\varphi + \rho_0\vartheta + \rho_0\wp\right) + \\ +\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{v}}\left(\rho_{0q}\mathbf{A} + \rho_0\mathbf{U} + \rho_0\mathbf{\Pi}\right) \end{bmatrix} \cdot\mathbf{v}\,u^0\sqrt{-g}\,dx^1dx^2dx^3 + \int_V \begin{pmatrix} \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} + \frac{c^2}{16\pi\eta}u_{\mu\nu}u^{\mu\nu} + \\ \frac{c^2}{16\pi\sigma}f_{\mu\nu}f^{\mu\nu} - \frac{1}{2}c\,k\,R \end{pmatrix} + \sqrt{-g}\,dx^1dx^2dx^3 + \sum_{n=1}^{N}\left(\mathbf{v}_n\cdot\frac{\partial L_f'}{\partial \mathbf{v}_n}\right). \qquad (69)$$

By solving equation (66), we can determine the metric tensor $g_{\mu\nu}$, then calculate the scalar curvature $R$ and, with its help, find the system's energy $E_i'$ (69) in matter. However, there still remains the problem of gauging the undetermined coefficients in the metric tensor in such a way that $R$ correctly and uniquely defines the energy in (69). Usually. in general relativity, the metric tensor is defined taking into account the fact that in the limit of a weak field the gravitational force transforms into the Newtonian force of gravitation. But in the general case, this may not be enough for the value $R$ obtained through such metric tensor to exactly satisfy the expression for energy (69).

We can avoid this problem in the following way. Let us suppose that the theory of vector fields developed in [1] is valid just like the general relativity is. Then we can equate the energy of general relativity (69) to the corresponding energy in the theory of vector fields.

For the part of the Lagrangian $L_f$ associated with $\mathcal{L}_f$ (3), under the gauge condition $R = 2\Lambda$, which is applied to vector fields for the purpose of calculating energy and momentum in continuously distributed matter, we can write

$$L_f = \int_V \mathcal{L}_f\sqrt{-g}\,dx^1dx^2dx^3 = \int_V \begin{pmatrix} -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} + \frac{c^2}{16\pi G}\Phi_{\mu\nu}\Phi^{\mu\nu} \\ -\frac{c^2}{16\pi\eta}u_{\mu\nu}u^{\mu\nu} - \frac{c^2}{16\pi\sigma}f_{\mu\nu}f^{\mu\nu} \end{pmatrix} \sqrt{-g}\,dx^1dx^2dx^3. \qquad (70)$$

If we substitute Lagrangian density $\mathcal{L} = \mathcal{L}_p + \mathcal{L}_f$ (3) into (19) under condition $R = 2\Lambda$ and $L_f$ (70), we obtain the energy $E_i$ for vector fields inside matter [8], [32]

$$E_i = \frac{1}{c}\int_V \begin{bmatrix} -\frac{\partial}{\partial \mathbf{v}}\begin{pmatrix}\rho_{0q}\varphi + \rho_0\psi \\ +\rho_0\vartheta + \rho_0\wp\end{pmatrix} + \\ +\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{v}}\begin{pmatrix}\rho_{0q}\mathbf{A} + \rho_0\mathbf{D} \\ +\rho_0\mathbf{U} + \rho_0\mathbf{\Pi}\end{pmatrix} \end{bmatrix} \cdot\mathbf{v}\,u^0\sqrt{-g}\,dx^1dx^2dx^3 + \int_V \begin{pmatrix} \frac{u^0}{c}\left(\rho_{0q}\varphi + \rho_0\psi + \rho_0\vartheta + \rho_0\wp\right) + \\ +\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - \frac{c^2}{16\pi G}\Phi_{\mu\nu}\Phi^{\mu\nu} \\ +\frac{c^2}{16\pi\eta}u_{\mu\nu}u^{\mu\nu} + \frac{c^2}{16\pi\sigma}f_{\mu\nu}f^{\mu\nu} \end{pmatrix} \sqrt{-g}\,dx^1dx^2dx^3 + \sum_{n=1}^{N}\left[\mathbf{v}_n\cdot\frac{\partial}{\partial \mathbf{v}_n}\int_V \begin{pmatrix} -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} \\ +\frac{c^2}{16\pi G}\Phi_{\mu\nu}\Phi^{\mu\nu} - \\ -\frac{c^2}{16\pi\eta}u_{\mu\nu}u^{\mu\nu} \\ -\frac{c^2}{16\pi\sigma}f_{\mu\nu}f^{\mu\nu} \end{pmatrix} \sqrt{-g}\,dx^1dx^2dx^3\right]. \qquad (71)$$

Let's equate energy $E_i'$ (69) and energy $E_i$ (71):

$$E_i' = E_i\,. \qquad (72)$$

The equality of energies in (72) allows us to impose additional conditions on the values of uncertain coefficients in the metric tensor in matter and on the value of scalar curvature $R$ present in energy (69). In this case, one should take into account the difference between metric tensors in GTR$^m$ and in the theory of vector fields, which follows from the difference in equations for metric. This leads to the fact that the tensors $F^{\mu\nu}$, $u^{\mu\nu}$ and $f^{\mu\nu}$ in (69), depending on the metric tensor, differ from the same tensors in (71). Therefore, similar terms in the left and right sides of (72), associated with tensors, cannot cancel with each other.

Let us now consider the situation outside of matter, where, according to general relativity, there is only an electromagnetic field and a metric field, and four-currents are equal to zero. In this case, the equation for metric (64) is simplified

$$2ckR_{\mu\nu} - ckRg_{\mu\nu} = -2ck\Lambda g_{\mu\nu} - W_{\mu\nu}. \qquad (73)$$

Contraction of equation (73) with the metric tensor $g^{\mu\nu}$ leads to relation $R = 4\Lambda$, so that if $\Lambda$ is constant, then the scalar curvature $R$ would also be constant. Substitution $R = 4\Lambda$ into (73) taking into account the equality $2ck = -\frac{c^4}{8\pi G\beta}$ leads to equation for metric outside of matter

$$R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} = \frac{8\pi G\beta}{c^4}W_{\mu\nu}. \qquad (74)$$

Outside matter, the mass density $\rho_0$ and charge density $\rho_{0q}$ are equal to zero, and expressions (62-63), (68) have the following form

$$\mathcal{L}'_p = 0,$$

$$\mathcal{L}'_f = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} + ckR - 2ck\Lambda. \qquad (75)$$

$$L'_f = \int_V \mathcal{L}'_f \sqrt{-g}\, dx^1 dx^2 dx^3 = \int_V \begin{pmatrix} -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} + \\ ckR - 2ck\Lambda \end{pmatrix} \sqrt{-g}\, dx^1 dx^2 dx^3. \qquad (76)$$

Taking into account the relation $R = 4\Lambda$ in (76), the expression for energy outside matter in GTR$^m$ version in (67) is also simplified

$$E'_o = \int_V \left( \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}ckR \right) \sqrt{-g}\, dx^1 dx^2 dx^3 + \sum_{n=1}^{N} \left[ \mathbf{v}_n \cdot \frac{\partial}{\partial \mathbf{v}_n} \int_V \begin{pmatrix} -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} \\ +\frac{1}{2}ckR \end{pmatrix} \sqrt{-g}\, dx^1 dx^2 dx^3 \right]. \qquad (77)$$

For vector fields outside matter, there are only electromagnetic and gravitational fields, and the relation $R = 4\Lambda = 0$ according to [1], [44] is valid. Instead of (71), the energy becomes equal to the following expression:

$$E_o = \int_V \begin{pmatrix} \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} \\ -\frac{c^2}{16\pi G}\Phi_{\mu\nu}\Phi^{\mu\nu} \end{pmatrix} \sqrt{-g}\, dx^1 dx^2 dx^3 + \sum_{n=1}^{N} \left[ \mathbf{v}_n \cdot \frac{\partial}{\partial \mathbf{v}_n} \int_V \begin{pmatrix} -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} \\ +\frac{c^2}{16\pi G}\Phi_{\mu\nu}\Phi^{\mu\nu} \end{pmatrix} \sqrt{-g}\, dx^1 dx^2 dx^3 \right]. \qquad (78)$$

By solving equation (74), one can find expressions for the metric tensor and scalar curvature $R$. Equality of energies in (77) and in (78) in the form

$$E'_o = E_o. \qquad (79)$$

makes it possible to clarify the value of scalar curvature $R$ in energy (77), as well as to clarify the values of uncertain coefficients in the metric tensor outside matter. In addition, the equality of internal and external metrics on the surface of massive body also allows us to more precisely define the undetermined coefficients in the metric tensor.

In a similar way as with the energy we can proceed with the system's momentum. A comparison of energies and momentums in OTO$^m$ and in vector fields, taking into account the formulas for momentum in Appendix B, leads to two relations (72) and (B8) for $R$ in matter, and to two relations (79) and (B11) for $R$ outside matter. After clarifying the value $R$ in OTO$^m$, it becomes possible to use formulas for energy in matter (69) and beyond matter (77). At the same time, according to (B7) and (B10) in Appendix B, the formulas for momentum in OTO$^m$, respectively, in matter and outside matter have the following form

$$\mathbf{P}'_i = \sum_{n=1}^{N} \frac{\partial}{\partial \mathbf{v}_n} \int_V \begin{pmatrix} -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} \\ -\frac{c^2}{16\pi\eta}u_{\mu\nu}u^{\mu\nu} - \\ \frac{c^2}{16\pi\sigma}f_{\mu\nu}f^{\mu\nu} + \frac{1}{2}ckR \end{pmatrix} \sqrt{-g}\, dx^1 dx^2 dx^3. \qquad (80)$$

$$\mathbf{P}'_o = \sum_{n=1}^{N} \frac{\partial}{\partial \mathbf{v}_n} \int_V \begin{pmatrix} -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} \\ +\frac{1}{2}ckR \end{pmatrix} \sqrt{-g}\, dx^1 dx^2 dx^3. \qquad (81)$$

By summing the energies inside and outside matter (69) and (77), we find the energy $E$ of a physical system; similarly, the sum of the momenta inside and outside matter (80) and (81) gives the momentum $\mathbf{P}$ of the system.

After the energy $E$ and the momentum $\mathbf{P}$ are found, we can determine the four-momentum of the system, defined in [8] in the form $P_\mu = \left( \frac{E}{c}, -\mathbf{P} \right)$. The energy determined in the center-of-momentum frame represents the rest energy $E_0$, with the help of which the system's inertial mass in accordance with [32] is calculated as $\mathcal{M} = \frac{E_0}{c^2}$.

In vector field theory, the gravitational mass of a system is calculated after the gravitational tensor $\Phi_{\mu\nu}$ is found, the time components of which include the gravitational field strength $\mathbf{\Gamma}$. Near the surface of a spherical massive body, the strength $\mathbf{\Gamma}$, according to Newton's law, is equal to the free fall acceleration, found in terms of the gravitational mass of the body. Thus, a connection between $\Phi_{\mu\nu}$ and the gravitational mass appears. Since the methods for determining the inertial and gravitational masses are completely different, these masses can equal each other only approximately. As a result, the principle of equivalence of the inertial and gravitational masses, which contributed to the development of general relativity, is not satisfied for vector fields.

Let us now consider the equation of motion of typical particles in matter in GTR$^m$. Using the principle of least action for the Lagrangian density $\mathcal{L}' = \mathcal{L}'_p + \mathcal{L}'_f$ (62-63) and varying over four-currents, we arrive at the equation of motion, which differs from the equation of motion for vector fields in [1] only in the absence of a gravitational term

$$J^\sigma u_{\sigma\mu} + J^\sigma f_{\sigma\mu} + j^\sigma F_{\sigma\mu} = 0. \qquad (82)$$

Equation (82) is also obtained from the expression $\nabla^\nu T'_{\mu\nu} = 0$, where the total stress-energy tensor of non-gravitational fields is presented in (66) in the form $T'_{\mu\nu} = W_{\mu\nu} + B_{\mu\nu} + P_{\mu\nu}$.

Let us express the tensors of acceleration field and pressure field in terms of the fields' four-potentials and substitute them into (82)

$$u_{\sigma\mu} = \nabla_\sigma U_\mu - \nabla_\mu U_\sigma,$$

$$f_{\sigma\mu} = \nabla_\sigma \pi_\mu - \nabla_\mu \pi_\sigma,$$

$$\begin{aligned} & J^\sigma \nabla_\sigma U_\mu - J^\sigma \nabla_\mu U_\sigma \\ & + J^\sigma \nabla_\sigma \pi_\mu - J^\sigma \nabla_\mu \pi_\sigma + j^\sigma F_{\sigma\mu} = 0 \end{aligned}. \qquad (83)$$

We apply (83) to a relativistic uniform system of a spherical shape with chaotically moving particles, which is kept in equilibrium by its proper fields. The root-mean-square velocity of the particles' motion in such a system depends only on the radius and reaches its maximum at the center [24], [29]. If the particles do not have proper vector potentials in the comoving reference frames of these particles, then, due to the chaotic character of the particles' motion in such a system, the global vector potentials of all the fields would be equal to zero. This leads to the absence of solenoidal vectors of the fields, similar to the magnetic field in the case of an electromagnetic field. To a first approximation, for the four-potential of acceleration field, describing the motion of an arbitrary typical particle of the system, the following relation holds true: $U_\mu = \left( \frac{\vartheta}{c}, -\mathbf{U} \right) = u_\mu$, where $u_\mu$ is the four-velocity of the particle, $\vartheta$ and $\mathbf{U}$ represent the scalar and vector potentials of the acceleration field, respectively. In the same approximation, the four-potential of the particle pressure field will equal $\pi_\mu = \left( \frac{\wp}{c}, -\mathbf{\Pi} \right) = \frac{\mathbb{P}_0}{\rho_0 c^2} u_\mu$ [45]. Let us substitute these potentials into (83)

$$\begin{aligned} & J^\sigma \nabla_\sigma u_\mu - J^\sigma \nabla_\mu u_\sigma + \\ & J^\sigma \nabla_\sigma \left( \frac{\mathbb{P}_0}{\rho_0 c^2} u_\mu \right) - J^\sigma \nabla_\mu \left( \frac{\mathbb{P}_0}{\rho_0 c^2} u_\sigma \right) + . \\ & j^\sigma F_{\sigma\mu} = 0 \end{aligned} \qquad (84)$$

Next, we use the continuity equation $\nabla_\sigma J^\sigma = 0$ and the following obvious relations

$$\begin{aligned} & J^\sigma \nabla_\sigma u_\mu = \nabla_\sigma \left( u_\mu J^\sigma \right) \\ & -u_\mu \nabla_\sigma J^\sigma = \nabla_\sigma \left( \rho_0 u_\mu u^\sigma \right) \end{aligned},$$

$$J^\sigma \nabla_\mu u_\sigma = \rho_0 u^\sigma \nabla_\mu u_\sigma = 0,$$

$$\begin{aligned} & J^\sigma \nabla_\sigma \left( \frac{\mathbb{P}_0}{\rho_0 c^2} u_\mu \right) = \nabla_\sigma \left( \frac{\mathbb{P}_0}{\rho_0 c^2} u_\mu J^\sigma \right) \\ & - \frac{\mathbb{P}_0}{\rho_0 c^2} u_\mu \nabla_\sigma J^\sigma = \nabla_\sigma \left( \frac{\mathbb{P}_0}{c^2} u_\mu u^\sigma \right) \end{aligned},$$

$$\begin{aligned} & J^\sigma \nabla_\mu \left( \frac{\mathbb{P}_0}{\rho_0 c^2} u_\sigma \right) = \frac{\mathbb{P}_0}{\rho_0 c^2} J^\sigma \nabla_\mu u_\sigma + \\ & u_\sigma J^\sigma \nabla_\mu \left( \frac{\mathbb{P}_0}{\rho_0 c^2} \right) = \rho_0 \nabla_\mu \left( \frac{\mathbb{P}_0}{\rho_0} \right) \end{aligned}. \qquad (85)$$

Taking into account (85), the equation (84) is written as follows

$$\begin{aligned} & \nabla_\sigma \left[ \left( \rho_0 + \frac{\mathbb{P}_0}{c^2} \right) u_\mu u^\sigma \right] = \\ & \rho_0 \nabla_\mu \left( \frac{\mathbb{P}_0}{\rho_0} \right) - j^\sigma F_{\sigma\mu} \end{aligned}. \qquad (86)$$

In relativistic uniform system, the equality $\nabla_\mu \rho_0 = \partial_\mu \rho_0 = \left( \frac{1}{c}\frac{\partial \rho_0}{\partial t}, \nabla \rho_0 \right) = 0$ holds true; the invariant mass density $\rho_0 = const$, that is, the density $\rho_0$ does not depend on either time or coordinates and is the same for all the particles in the system. In such a physical system $\rho_0 \nabla_\mu \left( \frac{\mathbb{P}_0}{\rho_0} \right) = \nabla_\mu \mathbb{P}_0$, and we can see that equation (86) coincides with the equation of motion (29) in general relativity when taking scalar pressure $\mathbb{P}_0$ into account.

On the other hand, as we indicated in Section 2, the equation of motion (29) of general relativity will be consistent with the principle of least action and (27) if $u_\nu \nabla^\nu \rho_0 = \frac{d\rho_0}{d\tau} = 0$; that is, the mass density $\rho_0$ must be constant in the comoving reference frame of each matter element. Since in this case $\frac{d\rho_0}{d\tau} = \frac{dt}{d\tau}\frac{d\rho_0}{dt} = \frac{u^0}{c}\frac{d\rho_0}{dt} = 0$, where $u^0$ is the time component of the four-velocity, then the condition $\frac{d\rho_0}{dt} = \frac{\partial \rho_0}{\partial t} + \mathbf{v} \cdot \nabla \rho_0 = 0$ must also be satisfied. All this is satisfied by the equality $\rho_0 = const$ for the relativistic uniform system. Thus, within the framework of general relativity, calculations of equation of motion of matter inside massive objects, such as compact stars, can be performed with condition $\rho_0 = const$. In all the other cases, for greater accuracy, it is better to use not equation (29), but rather the equation of motion in the form of (82-83), where the field tensors are found through the corresponding field equations.

Let us apply the covariant derivative $\nabla^\nu$ to both sides of equation (64) for the metric. On the left-hand side, we obtain zero as a consequence of the properties of the Einstein tensor. The right-hand side of (64) contains the total stress-energy tensor $T'_{\mu\nu} = W_{\mu\nu} + B_{\mu\nu} + P_{\mu\nu}$ of the three fields, for which the following equations hold

$$\nabla^\nu W_{\mu\nu} = j^\sigma F_{\sigma\mu}, \nabla^\nu B_{\mu\nu} = J^\sigma u_{\sigma\mu},$$

$$\nabla^\nu P_{\mu\nu} = J^\sigma f_{\sigma\mu}. \qquad (87)$$

Taking (87) into account, the following equation follows from (64)

$$\begin{aligned} &\nabla_\mu \left( A_\alpha j^\alpha + U_\alpha J^\alpha + \pi_\alpha J^\alpha + 2ck\Lambda \right) \\ &+ J^\sigma u_{\sigma\mu} + J^\sigma f_{\sigma\mu} + j^\sigma F_{\sigma\mu} = 0 \end{aligned}. \qquad (88)$$

If we take into account equation of motion (82) in (88), then using (65) we obtain for scalar curvature:

$$\nabla_\mu \begin{pmatrix} A_\alpha j^\alpha + U_\alpha J^\alpha \\ + \pi_\alpha J^\alpha + 2ck\Lambda \end{pmatrix} = \frac{ck}{2} \nabla_\mu R = 0. \qquad (89)$$

Condition (89) imposes an additional limitation on the quantity $R$ in matter in GTR$^m$.

### 3.5. Discussion of GTR$^m$ version

The GTR$^m$ version presented in the previous Section is more accurate than standard general relativity due to the use of vector acceleration field and vector pressure field instead of corresponding scalar fields. Indeed, it is difficult to directly include the scalar pressure $\mathbb{P}_0$ in Lagrangian density because we need to make additional assumptions about the variation $\delta \mathbb{P}_0$ to apply it into the principle of least action. As a result, the equation of motion (29) of general relativity is not derived from the principle of least action itself, but rather by equating the divergence of stress-energy tensor to zero in the form $\nabla^\nu T_{\mu\nu} = 0$.

However, from the standpoint of Lagrangian formalism, derivation of equation of motion from the principle of least action is preferable and necessary for completeness of the theory. The use of scalar function $K(J^\mu, g_{\mu\nu})$ allows us to derive the equation of motion (27) and to show that, on the condition that $\nabla_\mu \rho_0 = 0$, this function actually becomes equal to pressure $\mathbb{P}_0$ since in this case $K = \frac{\mathbb{P}_0 \sqrt{J^\mu J^\nu g_{\mu\nu}}}{\rho_0 c} = \mathbb{P}_0$. The problem of this approach is associated with the need to define a precise expression for the function $K(J^\mu, g_{\mu\nu})$ in general case, which requires solving the system of equations (45), (47) and (49-50) in GTR$^2$ version.

Achieving greater accuracy in GTR$^m$ version is possible due to the additional terms in Lagrangian density $\mathcal{L}' = \mathcal{L}'_p + \mathcal{L}'_f$ (62-63), which include the tensor invariants $u_{\mu\nu} u^{\mu\nu}$ of acceleration field and $f_{\mu\nu} f^{\mu\nu}$ of pressure field. The addition of these terms leads to the emergence of independent equations of corresponding fields and allows us to quickly find all the characteristics of these fields in standard form.

Substitution of gravitation by the spacetime curvature, and reduction of physical force of body attraction to geometry were fully justified in general relativity for the case of motion of small test bodies near massive objects, as happens in the case of motion of planets and rays of light near the Sun. However, in obtaining solutions for the case of continuous matter with pressure and electromagnetic field, as was shown above, we face various problems. One of these problems is related to the system's energy and momentum, and the other is related to the ambiguity of solutions for the metric. The fact is that the energy and momentum cannot be determined without considering the contribution of gravitational field.

However, since gravitation is included in the metric, it is first necessary to solve the equations for metric (66) and (74) inside and outside matter, to find the metric tensor and scalar curvature $R$, and through them evaluate the contribution of gravitation in energy and momentum. If we use the Lagrangian formalism, energy and momentum inside matter in GTR$^m$ can be found using formulas (69) and (80), respectively, and energy and momentum outside matter can be found using formulas (77) and (81).

With this method, ambiguity arises in the definition of energy and momentum, since the solutions for the metric tensor contain undefined coefficients resulting from the integration of the equations. To avoid such ambiguity, we proposed to use energy and momentum, calculated in the theory of vector fields, as auxiliary quantities. Comparison of these quantities with the energy in GTR$^m$ in (72), (79), and momentum in (B8) and (B11) in Appendix B makes it possible to clarify the values $R$ inside and outside matter and thereby unambiguously determine the energy and momentum of the system. However, it should be noted that the scalar curvature inside matter must simultaneously satisfy both equality for energy (72) and equality for momentum (B8). Similarly, the scalar curvature outside matter must simultaneously satisfy both the equality for energy (79) and the equality for momentum (B11). In this case, at the boundary of a body, the scalar curvature inside matter must be equal to the scalar curvature outside matter.

The proposed approach is a consequence of Lagrangian formalism with respect to energy and momentum. Therefore, this approach has an advantage over standard general relativity approach, where the energy and momentum are defined in terms of volume integral of time components of stress-energy tensor summed with gravitational pseudotensor components.

Let us consider, as an example, the symmetric Landau–Lifshitz pseudotensor of gravitational field $t^{\mu\nu}$ [38], for which, in view of stress-energy tensor of matter and non-gravitational fields, the coefficient $\varkappa = \frac{8\pi G}{c^4}$ and the cosmological constant $\Lambda_{GR}$, the following equation holds

$$\partial_\nu \left[ (-g) \left( T^{\mu\nu} + t^{\mu\nu} - \frac{\Lambda_{GR}}{\varkappa} g^{\mu\nu} \right) \right] = 0 . \qquad (90)$$

Integrating (90) over infinite volume gives the following

$$P^\mu_{LL} = \frac{1}{c} \int (-g) \left( T^{\mu 0} + t^{\mu 0} - \frac{\Lambda_{GR}}{\varkappa} g^{\mu\nu} \right) dx^1 dx^2 dx^3 . \qquad (91)$$

It is asserted that the integral vector $P^\mu_{LL}$ (91) represents the four-momentum of a system.

We noted some drawbacks of standard general relativity approach in Section 3.3, while discussing GTR$^1$ and GTR$^2$. We can add that an additional drawback is the lack of mathematical proof that volume integral (91) of time components of stress-energy tensor summed with gravitational pseudotensor components precisely gives the four-momentum of a system, not any other value. At least such a proof does not follow from the Lagrangian formalism [8].

Indeed, treatment of $P^\mu_{LL}$ (91) as a four-momentum starts with the fact that stress-energy tensor $T^{\mu\nu}$ is expressed through the stress-energy tensor $\tau_{\mu\nu}$ (16) in sum with the stress-energy tensor $W_{\mu\nu}$ (15) of electromagnetic field. Next, the weak gravitational field approximation is used when we can assume that $t^{\mu\nu} \approx 0$ in comparison with $T^{\mu\nu}$. Then, to a first approximation, the value $P^\mu_{LL}$ is close to the value of the system's four-momentum. Hence, it is assumed that in the general case, $P^\mu_{LL}$ is also the four-momentum.

In response to such argumentation, we would like to remember that the equation of motion (29) in standard general relativity is consistent with the equation of motion (27), derived from the principle of least action, and is valid only on the condition that $u_\nu \nabla^\nu \rho_0 = \frac{d\rho_0}{d\tau} = 0$.

This condition is equivalent to the fact that a relativistic uniform system is always under consideration. As was shown in [24], [29], equilibrium in a relativistic uniform system reduces to equilibrium in gravitational and electromagnetic fields.in acceleration field and in pressure field. If we consider the situation not from the standpoint of general relativity, but from the standpoint of vector fields, then instead of (90) we must proceed from the equation of matter's motion in the form

$$\nabla_\nu T^{\mu\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\nu_{\rho\nu} T^{\mu\rho} + \Gamma^\mu_{\rho\nu} T^{\rho\nu} = \frac{1}{\sqrt{-g}} \frac{\partial \left( \sqrt{-g} T^{\mu\nu} \right)}{\partial x^\nu} + \Gamma^\mu_{\rho\nu} T^{\rho\nu} = 0 , \qquad (92)$$

In (92), stress-energy tensor $T^{\mu\nu} = W^{\mu\nu} + U^{\mu\nu} + B^{\mu\nu} + P^{\mu\nu}$ includes stress-energy tensors of all four fields; therefore, there is no need for any gravitational field pseudotensor.

Let's choose a reference frame in which the Christoffel symbols $\Gamma^\mu_{\rho\nu}$ in some element of matter are zero. In this case, multiplying (92) by the element of covariant volume $\sqrt{-g}\, dx^0 dx^1 dx^2 dx^3$ and integrating over the four-dimensional volume of this element, taking into account the divergence theorem, we have:

$$\int \frac{\partial\left(\sqrt{-g}T^{\mu\nu}\right)}{\partial x^{\nu}} dx^0 dx^1 dx^2 dx^3 =$$
$$\int T^{\mu 0}\sqrt{-g}\, dx^1 dx^2 dx^3 +$$
$$\int\left[\int T^{\mu 1}\sqrt{-g}\, dx^2 dx^3\right] dx^0 + \qquad (93)$$
$$+\int\left[\int T^{\mu 2}\sqrt{-g}\, dx^1 dx^3\right] dx^0 +$$
$$\int\left[\int T^{\mu 3}\sqrt{-g}\, dx^1 dx^2\right] dx^0 \approx 0.$$

Let's make the notation:

$$I^{\mu} = \frac{1}{c}\int T^{\mu 0}\sqrt{-g}\, dx^1 dx^2 dx^3 . \qquad (94)$$

$$\int T^{\mu 1}\sqrt{-g}\, dx^2 dx^3 + \int T^{\mu 2}\sqrt{-g}\, dx^1 dx^3$$
$$+\int T^{\mu 3}\sqrt{-g}\, dx^1 dx^2 = \oiint_S T^{\mu k} n_k \sqrt{-g}\, dS . \qquad (95)$$

In (95), the sum of the three integrals is a surface integral over a two-dimensional surface $S$, surrounding the volume element, $n_k$ is a unit vector perpendicular to the surface $S$ and directed outward, $k = 1,2,3$. Substituting (94-95) into (93) and differentiating by variable $x^0 = ct$ , we find

$$\frac{dI^{\mu}}{dt} + \oiint_S T^{\mu k} n_k \sqrt{-g}\, dS \approx 0 . \qquad (96)$$

The smaller the volume element in question is selected, the more precisely expressions (93) and (96) tend to zero. At $\mu = 0$ (96) describes the generalized Poynting theorem in integral form, according to which energy fluxes flowing into a certain volume increase the energy of fields in this volume [28]. When $\mu = 1,2,3$ the values $T^{\mu k}$ taken with a minus sign are components of a three-dimensional stress tensor. In this case (96) can be considered as integral equations for the rates of change of energy fluxes in an element of matter. Such changes in energy fluxes are caused by forces acting on the element of matter from the fields.

Suppose that the volume element in question is in such an equilibrium state that there are no energy fluxes through its surface or the fluxes are on average zero. In this case, according to (96), $I^{\mu}$ becomes a certain constant in time, $I^{\mu} = const$ .

It is not difficult to verify that at equilibrium the integral over the three-dimensional volume in matter in (94) vanishes [8], [28]. This is a consequence of equation of motion in the form $\nabla^{\nu} T_{\mu\nu} = 0$ , that is, the consequence of balance of all the forces in matter at equilibrium. If in (96) the volume element is taken not in matter, but outside it, then in (96) only the total energy of gravitational and electromagnetic fields outside matter and the fluxes of these energies remain. It turns out that the integral vector $I^{\mu}$ does not set the four-momentum of the element of matter, and even more so does not set the four-momentum of entire system consisting of many particles and fields. Instead, the vector $I^{\mu}$ shows that in each volume element of a closed equilibrium system, another value associated with the energy of fields must be preserved.

To obtain the vector $I^{\mu}$, we had to use the weak field approximation by choosing a suitable reference frame in which the Christoffel symbols in the volume element in question become zero. But in the general case $I^{\mu}$ turns out to be a four-dimensional pseudovector, since the equation of motion $\nabla^{\nu} T_{\mu\nu} = 0$ in covariant form does not integrate over four-dimensional volume and does not give a true four-vector.

The presented picture shows that the integral vector $P^{\mu}_{LL}$ in (91), like $I^{\mu}$ in (94), is not the system's four-momentum, but an integral pseudovector. In this case, there is no other way in general relativity to find the energy and momentum, than to use the formulas derived from the Lagrangian mechanism and presented above.

We analyzed in more detail the problem of four-momentum and integral vector in general relativity and in theory of vector fields in [8] and [27] where references were also provided to papers showing inadequacy of general relativity approach for defining of energy and momentum. For example, in [37] indicated that the energy of a closed system in general relativity is either not conserved or depends on the choice of reference frame. It can be seen from (91) that tensor $T^{\mu\nu}$ and pseudotensor $t^{\mu\nu}$ have different transformation laws; therefore, the system's inertial mass, which should be obtained from $P^{\mu}_{LL}$, will not be the same in different reference frames. This is confirmed in [46], which also indicates the inequality of inertial and gravitational masses of a physical system in general relativity. Moreover, according to [47], the principle of correspondence does not hold in general relativity.

In addition, even if $t^{\mu\nu}$ is a tensor, $P^{\mu}_{LL}$ cannot be an actual four-vector. This follows from the fact that the right-hand side of (91) contains the time tensor components that are transformed into another reference frame in a different way than the components of a four-vector should be transformed [28]. The difference in transformation of tensor components and four-vector components leads to the so-called 4/3 problem for a moving body, when the mass-energy in volume integral of the time component of stress-energy tensor for electromagnetic or gravitational fields is not equal to the mass-energy in integral of the space components of this tensor.

From a philosophical standpoint, noncoincidence of four-momentum and integral pseudovector $I^{\mu}$ in (94) is associated with the duality of matter and field and with the difference in their definitions in terms of four-currents and field tensors, respectively. The conservation of four-momentum in a closed system is associated with the conservation of energy and momentum of the matter's particles that generate fields and act on each other through these fields. At the same time, the conservation of integral pseudovector $I^{\mu}$ leads only to conservation of energy and energy flux of fields in the system.

## 4. Conclusions

In order to covariantly describe the pressure effect, we introduced the scalar function $K(J^{\mu}, g_{\mu\nu})$, which depend on the four-current $J^{\mu}$ and the metric tensor $g_{\mu\nu}$, into the Lagrangian density (1) of general relativity. Next, we found the equation for metric (17), derived the formulas for energy (20) and momentum (22), obtained equations of motion (27-29) and in (30) related the function $K(J^{\mu}, g_{\mu\nu})$ to scalar isotropic pressure $\mathbb{P}_0$ in matter.

With this in mind, in Section 3.1 we arrived at GTR$^1$ version, which is the closest to the standard general relativity, and in Section 3.2 at GTR$^2$ version, which was fully derived from the principle of least action. One of the results is that the equation of motion (29) in GTR$^1$ is consistent with equation (27) only on the condition that $u_{\nu}\nabla^{\nu}\rho_0 = \frac{d\rho_0}{d\tau} = 0$. This means that general relativity can be used to study relativistic uniform systems, where $\nabla^{\nu}\rho_0 = 0$, but it may be inaccurate in general cases.

The situation can be improved by using GTR$^2$; however, the analysis of both versions of general relativity in Section 3.3 revealed the presence of other notable drawbacks. For example, in general relativity, the expression of continuity equation (59) differs from the standard expression $\nabla_{\mu}\left(\rho_0 u^{\mu}\right) = 0$. As we show when deducing from the principle of least action in OTO$^2$ version, the equation of motion (27) agrees with (29) only under the condition $\nabla_{\mu}u^{\mu} = 0$. If we accept both the conditions $\nabla^{\nu}\rho_0 = 0$ and $\nabla_{\mu}u^{\mu} = 0$ in general relativity, then only in this case (59) passes into standard continuity equation $\nabla_{\mu}\left(\rho_0 u^{\mu}\right) = u^{\mu}\nabla_{\mu}\rho_0 + \rho_0\nabla_{\mu}u^{\mu} = 0$.

With the help of (61), we explain the meaning of dark energy, which emerges from the cosmological model of general relativity, and is expressed as $\left(2\rho_0 c^2 + A_{\beta}\, j^{\beta} + \mathbb{P}_0\right) g_{\mu\nu}$ in terms of rest energy density of cosmological matter $\rho_0 c^2$, energy density of particles' four-current $A_{\beta}\, j^{\beta}$ and pressure $\mathbb{P}_0$ in matter. In this case, the dark energy emerges because the equation for the metric (36) in general relativity is not derived from the principle of least action; and according to (37) $\Lambda_{GR}$ turns out to be a scalar function and is not a real cosmological constant.

In Section 3.4 we present modernized general theory of relativity, which we designate GTR$^m$. Unlike in standard general relativity, in GTR$^m$ acceleration field and pressure field are considered not as scalar fields, but as vector fields. Thus, for these fields it becomes possible to write their own equations and to find four-potentials, tensors and stress-energy tensors at a given mass four-current. This means, for example, that we no longer need to choose a possible equation for the state of matter that relates the pressure and the mass density; – for this reason. it suffices to solve standard differential equation for pressure field. The gravitational field, according to general relativity approach, included in metric field, which is geometric in nature. Thus, in OTO$^m$, gravitation is still reduced to spacetime curvature.

To determine the energy and momentum as easily as possible, in the GTR$^m$ version we suggest using four-potential $D_{\mu}$ and gravitational field tensor $\Phi_{\mu\nu}$ as auxiliary quantities, taken from the theory of vector fields. With the help of $D_{\mu}$ and $\Phi_{\mu\nu}$, one can calculate for vector fields the system's energy inside and outside matter using formulas (71) and (78), and the momentum of the system inside and outside matter using formulas (B1) and (B9) in Appendix B. In this case, conditions (72), (79), (B8) and (B11) in Appendix B make it possible to unambiguously gauge both the components of metric tensor, as well as the energy and momentum in GTR$^m$ version.

The equation of motion (83) in GTR$^m$ version is fully written in terms of four-potentials and tensors of the fields represented in a system. In the limit of relativistic uniform model, equation (83) becomes equal to (86) and exactly transforms into the equation of motion (29) of standard general relativity. Thus, the GTR$^m$ version can be considered an improved version of general relativity in many respects. On the other hand, the GTR$^m$ version is much closer to the theory of vector fields than to standard general relativity, which can be seen from comparison of Lagrangian density $\mathcal{L}' = \mathcal{L}'_p + \mathcal{L}'_f$ (62-63) and Lagrangian density (3). The difference between these theories lies only in the fact that in theory of vector fields gravitational four-potential and gravitational field tensor are directly included in Lagrangian density.

The advantage of vector fields is that the equation of motion can be obtained and confirmed in two different ways – either from the principle of least action or from the equation $\nabla^{\nu}T_{\mu\nu} = 0$ [1], [8]. Another advantage is that in the formulas for energy and momentum, due to use of energy gauging with the help of cosmological constant $\Lambda$, we can eliminate scalar curvature $R$ and thus uniquely determine the formulas. In this case, the approach used in theory of vector fields is preferable to that used in general relativity, since it is based entirely on Lagrangian formalism [32].

## Appendix A

We proceed from the variation $\delta S_1$ containing variations $\delta x^\mu$

$$\delta S_1 = -\int_{t_1}^{t_2}\int_V \left\{ \begin{array}{l} A_\mu \frac{1}{\sqrt{-g}} \partial_\sigma \\ \left[ \sqrt{-g} \left( \begin{array}{l} j^\sigma \delta x^\mu \\ -j^\mu \delta x^\sigma \end{array} \right) \right] + \\ + \left( u_\mu + \frac{\partial K}{\partial J^\mu} \right) \nabla_\sigma \\ \left( J^\sigma \delta x^\mu - J^\mu \delta x^\sigma \right) \end{array} \right\} dt = 0. \quad \text{(A1)}$$
$$\sqrt{-g}\, dx^1 dx^2 dx^3$$

We transform by parts in (A1) the term with the mass four-current $J^\mu$

$$-\int_{t_1}^{t_2}\int_V \left( u_\mu + \frac{\partial K}{\partial J^\mu} \right) \nabla_\sigma \left( \begin{array}{l} J^\sigma \delta x^\mu \\ -J^\mu \delta x^\sigma \end{array} \right) dt =$$
$$\sqrt{-g}\, dx^1 dx^2 dx^3$$

$$= -\int_{t_1}^{t_2}\int_V \nabla_\sigma \left[ \begin{array}{l} \left( u_\mu + \frac{\partial K}{\partial J^\mu} \right) \\ \left( J^\sigma \delta x^\mu - J^\mu \delta x^\sigma \right) \end{array} \right] dt + \quad \text{(A2)}$$
$$\sqrt{-g}\, dx^1 dx^2 dx^3$$

$$+\int_{t_1}^{t_2}\int_V \left( \begin{array}{l} J^\sigma \delta x^\mu \\ -J^\mu \delta x^\sigma \end{array} \right) \nabla_\sigma \left( u_\mu + \frac{\partial K}{\partial J^\mu} \right) dt.$$
$$\sqrt{-g}\, dx^1 dx^2 dx^3$$

The covariant divergence of an arbitrary four-vector $B^\sigma$ can be expressed as follows

$$\nabla_\sigma B^\sigma = \frac{1}{\sqrt{-g}} \partial_\sigma \left( \sqrt{-g}\, B^\sigma \right). \quad \text{(A3)}$$

Taking (A3) into account, the first integral on the right side of A2) can be written as follows

$$-\int_{t_1}^{t_2}\int_V \nabla_\sigma \left[ \begin{array}{l} \left( u_\mu + \frac{\partial K}{\partial J^\mu} \right) \\ \left( J^\sigma \delta x^\mu - J^\mu \delta x^\sigma \right) \end{array} \right] dt =$$
$$\sqrt{-g}\, dx^1 dx^2 dx^3$$

$$= -\int_{t_1}^{t_2}\int_V \partial_\sigma \left[ \begin{array}{l} \sqrt{-g} \left( u_\mu + \frac{\partial K}{\partial J^\mu} \right) \\ \left( J^\sigma \delta x^\mu - J^\mu \delta x^\sigma \right) \end{array} \right] dt. \quad \text{(A4)}$$
$$dx^1 dx^2 dx^3$$

We now use the divergence theorem for the right-hand side of (A4), moving from integrating the divergence of a four-vector over a four-dimensional volume to integrating the corresponding four-vector over four three-dimensional hypersurfaces

$$-\int_{t_1}^{t_2}\int_V \partial_\sigma \left[ \begin{array}{l} \sqrt{-g} \left( u_\mu + \frac{\partial K}{\partial J^\mu} \right) \\ \left( J^\sigma \delta x^\mu - J^\mu \delta x^\sigma \right) \end{array} \right] dt =$$
$$dx^1 dx^2 dx^3$$

$$= -\frac{1}{c} \left[ \int_V \begin{array}{l} \left( u_\mu + \frac{\partial K}{\partial J^\mu} \right) \\ \left( J^\sigma \delta x^\mu - J^\mu \delta x^\sigma \right) \\ \sqrt{-g}\, dx^1 dx^2 dx^3 \end{array} \right]_{t_1}^{t_2} - \quad \text{(A5)}$$

$$-\int_{t_1}^{t_2}\oint_\Sigma \begin{array}{l} \left( u_\mu + \frac{\partial K}{\partial J^\mu} \right) \\ \left( J^\sigma \delta x^\mu - J^\mu \delta x^\sigma \right) \end{array} dt = 0.$$
$$n_j \sqrt{-g}\, d\Sigma$$

The three-dimensional unit vector $n_j$, where the index $j = 1,2,3$, represents an outward-directed normal vector to the two-dimensional surface $\Sigma$, surrounding moving physical system under consideration. The equality to zero in (A5) follows from the fact that the variations $\delta x^\mu$ at the time points $t_1$ and $t_2$ are equal to zero according to the condition of variation of action function. In addition, in the case of integration over the surface $\Sigma$, the variation $\delta x^\mu$ on this surface is also considered to equal zero.

According to (A4-A5), the first integral on the right side of (A2) is equal to zero. The second integral in (A2) is transformed as follows

$$\int_{t_1}^{t_2}\int_V \begin{array}{l} \left( J^\sigma \delta x^\mu - J^\mu \delta x^\sigma \right) \\ \nabla_\sigma \left( u_\mu + \frac{\partial K}{\partial J^\mu} \right) \sqrt{-g}\, dx^1 dx^2 dx^3 \end{array} dt =$$

$$= \int_{t_1}^{t_2}\int_V \left[ \begin{array}{l} J^\sigma \nabla_\sigma \left( u_\mu + \frac{\partial K}{\partial J^\mu} \right) \\ -J^\sigma \nabla_\mu \left( u_\sigma + \frac{\partial K}{\partial J^\sigma} \right) \end{array} \right] dt =$$
$$\delta x^\mu \sqrt{-g}\, dx^1 dx^2 dx^3$$

$$= \int_{t_1}^{t_2}\int_V \left[ \begin{array}{l} J^\sigma \nabla_\sigma \left( u_\mu + \frac{\partial K}{\partial J^\mu} \right) \\ -J^\sigma \nabla_\mu \frac{\partial K}{\partial J^\sigma} \end{array} \right] dt.$$
$$\delta x^\mu \sqrt{-g}\, dx^1 dx^2 dx^3 \quad \text{(A6)}$$

In (A6) we used the relation $J^{\sigma}\nabla_{\mu}u_{\sigma}=\rho_0 u^{\sigma}\nabla_{\mu}u_{\sigma}$ $=\frac{\rho_0}{2}\nabla_{\mu}\left(u^{\sigma}u_{\sigma}\right)=\frac{\rho_0}{2}\nabla_{\mu}c^2=0$. Consequently, (A1) is equivalent to the following

$$\delta S_1=-\int_{t_1}^{t_2}\int_V \left\{\begin{matrix} A_{\mu}\frac{1}{\sqrt{-g}}\partial_{\sigma} \\ \left[\sqrt{-g}\begin{pmatrix} j^{\sigma}\delta x^{\mu} \\ -j^{\mu}\delta x^{\sigma}\end{pmatrix}\right]- \\ -\begin{bmatrix} J^{\sigma}\nabla_{\sigma}\left(u_{\mu}+\frac{\partial K}{\partial J^{\mu}}\right) \\ -J^{\sigma}\nabla_{\mu}\frac{\partial K}{\partial J^{\sigma}}\end{bmatrix}\delta x^{\mu} \end{matrix}\right\} \sqrt{-g}\,dx^1dx^2dx^3\,dt=0. \quad (A7)$$

Let us transform the first integral in (A7)

$$-\int_{t_1}^{t_2}\int_V \left\{\begin{matrix} A_{\mu}\frac{1}{\sqrt{-g}}\partial_{\sigma} \\ \left[\sqrt{-g}\left(j^{\sigma}\delta x^{\mu}-j^{\mu}\delta x^{\sigma}\right)\right]\end{matrix}\right\} \sqrt{-g}\,dx^1dx^2dx^3\,dt=$$

$$=-\int_{t_1}^{t_2}\int_V \partial_{\sigma}\begin{bmatrix} A_{\mu}\sqrt{-g} \\ \left(j^{\sigma}\delta x^{\mu}-j^{\mu}\delta x^{\sigma}\right)\end{bmatrix} dx^1dx^2dx^3\,dt+ \quad (A8)$$

$$+\int_{t_1}^{t_2}\int_V \sqrt{-g}\begin{pmatrix} j^{\sigma}\delta x^{\mu} \\ -j^{\mu}\delta x^{\sigma}\end{pmatrix} \partial_{\sigma}A_{\mu}\,dx^1dx^2dx^3\,dt.$$

Taking into account the divergence theorem as in (A5), the first integral on the right side of (A8) is equal to zero. The second integral in (A8) is transformed as follows:

$$\int_{t_1}^{t_2}\int_V \sqrt{-g}\left(j^{\sigma}\delta x^{\mu}-j^{\mu}\delta x^{\sigma}\right)\partial_{\sigma}A_{\mu}\,dx^1dx^2dx^3\,dt=$$

$$=\int_{t_1}^{t_2}\int_V \begin{pmatrix} j^{\sigma}\delta x^{\mu}\partial_{\sigma}A_{\mu}- \\ j^{\mu}\delta x^{\sigma}\partial_{\sigma}A_{\mu}\end{pmatrix}\sqrt{-g}\,dx^1dx^2dx^3\,dt=$$

$$=\int_{t_1}^{t_2}\int_V \begin{pmatrix} j^{\sigma}\delta x^{\mu}\partial_{\sigma}A_{\mu}- \\ j^{\sigma}\delta x^{\mu}\partial_{\mu}A_{\sigma}\end{pmatrix}\sqrt{-g}\,dx^1dx^2dx^3\,dt= \quad (A9)$$

$$=\int_{t_1}^{t_2}\int_V j^{\sigma}\left(\partial_{\sigma}A_{\mu}-\partial_{\mu}A_{\sigma}\right)\delta x^{\mu}\sqrt{-g}\,dx^1dx^2dx^3\,dt=$$

$$\int_{t_1}^{t_2}\int_V j^{\sigma}F_{\sigma\mu}\delta x^{\mu}\sqrt{-g}\,dx^1dx^2dx^3dt.$$

From (A8-A9) it follows:

$$-\int_{t_1}^{t_2}\int_V \left\{\begin{matrix} A_{\mu}\frac{1}{\sqrt{-g}}\partial_{\sigma} \\ \left[\sqrt{-g}\begin{pmatrix} j^{\sigma}\delta x^{\mu}- \\ j^{\mu}\delta x^{\sigma}\end{pmatrix}\right]\end{matrix}\right\} \sqrt{-g}\,dx^1dx^2dx^3\,dt= \quad (A10)$$

$$=\int_{t_1}^{t_2}\int_V j^{\sigma}F_{\sigma\mu}\delta x^{\mu}\sqrt{-g}\,dx^1dx^2dx^3\,dt.$$

Taking (A10) into account, from (A7) we obtain the action variation and equation of motion

$$\delta S_1=\int_{t_1}^{t_2}\int_V \begin{bmatrix} j^{\sigma}F_{\sigma\mu} \\ +J^{\sigma}\nabla_{\sigma}\left(u_{\mu}+\frac{\partial K}{\partial J^{\mu}}\right) \\ -J^{\sigma}\nabla_{\mu}\frac{\partial K}{\partial J^{\sigma}}\end{bmatrix} \delta x^{\mu}\sqrt{-g}\,dx^1dx^2dx^3\,dt=0.$$

$$J^{\sigma}\nabla_{\sigma}\left(u_{\mu}+\frac{\partial K}{\partial J^{\mu}}\right)=J^{\sigma}\nabla_{\mu}\frac{\partial K}{\partial J^{\sigma}}-j^{\sigma}F_{\sigma\mu}. \quad (A11)$$

## Appendix B

For convenience, this appendix uses double numbering of formulas, indicating the corresponding formulas in text of the article.

If we substitute the Lagrangian density $\mathcal{L}=\mathcal{L}_p+\mathcal{L}_f$ (3) into (21) and take into account $L_f$ (70), we obtain the momentum of a system in theory of vector fields [8], [32]:

$$\mathbf{P}=\frac{1}{c}\int_V \left\{\begin{array}{l}\rho_{0q}\,\mathbf{A}+\rho_0\,\mathbf{D}+\\ \rho_0\,\mathbf{U}+\rho_0\,\mathbf{\Pi}-\dfrac{\partial}{\partial\mathbf{v}}\\ \begin{pmatrix}\rho_{0q}\,\varphi+\rho_0\psi+\\ \rho_0\vartheta+\rho_0\wp\end{pmatrix}+\\ +\left[\dfrac{\partial}{\partial\mathbf{v}}\begin{pmatrix}\rho_{0q}\,\mathbf{A}+\\ \rho_0\,\mathbf{D}+\rho_0\,\mathbf{U}\\ +\rho_0\,\mathbf{\Pi}\end{pmatrix}\right]\cdot\mathbf{v}\end{array}\right\}+ u^0\sqrt{-g}\,dx^1dx^2dx^3 +\sum_{n=1}^{N}\frac{\partial}{\partial\mathbf{v}_n}\int_V\begin{pmatrix}-\dfrac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}\\ +\dfrac{c^2}{16\pi G}\Phi_{\mu\nu}\Phi^{\mu\nu}\\ -\dfrac{c^2}{16\pi\eta}u_{\mu\nu}u^{\mu\nu}\\ -\dfrac{c^2}{16\pi\sigma}f_{\mu\nu}f^{\mu\nu}\end{pmatrix}\sqrt{-g}\,dx^1dx^2dx^3. \qquad \text{(B1)}$$

We now use the Lagrangian $\mathcal{L}'=\mathcal{L}'_p+\mathcal{L}'_f$ (62-63) and $L'_f$ (68), replacing in (21) $\mathcal{L}_p$ by $\mathcal{L}'_f$, $\mathcal{L}_f$ by $\mathcal{L}'_f$, and $L_f$ by $L'_f$. Thus, we find an expression for the momentum in GTR[m]

$$\mathbf{P}'=\frac{1}{c}\int_V\left[\begin{array}{l}\rho_{0q}\,\mathbf{A}+\rho_0\,\mathbf{U}+\rho_0\,\mathbf{\Pi}\\ -\dfrac{\partial}{\partial\mathbf{v}}\begin{pmatrix}\rho_{0q}\,\varphi+\rho_0\vartheta\\ +\rho_0\wp\end{pmatrix}+\\ +\mathbf{v}\cdot\dfrac{\partial}{\partial\mathbf{v}}\begin{pmatrix}\rho_{0q}\,\mathbf{A}+\rho_0\,\mathbf{U}\\ +\rho_0\,\mathbf{\Pi}\end{pmatrix}\end{array}\right]+ u^0\sqrt{-g}\,dx^1dx^2dx^3 +\sum_{n=1}^{N}\frac{\partial}{\partial\mathbf{v}_n}\int_V\begin{pmatrix}-\dfrac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}\\ -\dfrac{c^2}{16\pi\eta}u_{\mu\nu}u^{\mu\nu}\\ -\dfrac{c^2}{16\pi\sigma}f_{\mu\nu}f^{\mu\nu}\\ +c\,k\,R-2c\,k\,\Lambda\end{pmatrix}\sqrt{-g}\,dx^1dx^2dx^3. \qquad \text{(B2)}$$

In matter, the quantities $R$ and $\Lambda$ are related by relation (65), which allows us to express in (B2) $\Lambda$ through $R$

$$\mathbf{P}'_i=\frac{1}{c}\int_V\left[\begin{array}{l}\rho_{0q}\,\mathbf{A}+\rho_0\,\mathbf{U}+\rho_0\,\mathbf{\Pi}\\ -\dfrac{\partial}{\partial\mathbf{v}}\begin{pmatrix}\rho_{0q}\,\varphi\\ +\rho_0\vartheta\\ +\rho_0\wp\end{pmatrix}+\\ +\mathbf{v}\cdot\dfrac{\partial}{\partial\mathbf{v}}\begin{pmatrix}\rho_{0q}\,\mathbf{A}\\ +\rho_0\,\mathbf{U}\\ +\rho_0\,\mathbf{\Pi}\end{pmatrix}\end{array}\right]+ u^0\sqrt{-g}\,dx^1dx^2dx^3 +\sum_{n=1}^{N}\frac{\partial}{\partial\mathbf{v}_n}\int_V\begin{pmatrix}-\dfrac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}\\ -\dfrac{c^2}{16\pi\eta}u_{\mu\nu}u^{\mu\nu}\\ -\dfrac{c^2}{16\pi\sigma}f_{\mu\nu}f^{\mu\nu}\\ +\dfrac{1}{2}c\,k\,R+\\ +A_\alpha\,j^\alpha+U_\alpha\,J^\alpha\\ +\pi_\alpha\,J^\alpha\end{pmatrix}\sqrt{-g}\,dx^1dx^2dx^3. \qquad \text{(B3)}$$

We further use (62) in the form

$$A_\mu\,j^\mu+U_\mu\,J^\mu+\pi_\mu\,J^\mu =\frac{u^0}{c}\begin{pmatrix}\rho_{0q}\,\varphi-\rho_{0q}\,\mathbf{A}\cdot\mathbf{v}+\rho_0\vartheta\\ -\rho_0\,\mathbf{U}\cdot\mathbf{v}+\rho_0\wp-\rho_0\,\mathbf{\Pi}\cdot\mathbf{v}\end{pmatrix}, \qquad \text{(B4)}$$

as well as a relation from [35]:

$$\frac{dt}{d\tau}\sqrt{-g}\,dx^1dx^2dx^3 =\frac{u^0}{c}\sqrt{-g}\,dx^1dx^2dx^3=dV_0, \qquad \text{(B5)}$$

where $dV_0$ is differential of invariant proper volume of any particle of a continuously distributed matter.

Taking into account (B4- B5) we find:

$$\sum_{n=1}^{N}\frac{\partial}{\partial \mathbf{v}_n}\int_V \begin{pmatrix} A_\alpha\, j^{\alpha} + \\ U_\alpha\, J^{\alpha} + \pi_\alpha\, J^{\alpha} \end{pmatrix} = \sqrt{-g}\, dx^1 dx^2 dx^3$$

$$= \sum_{n=1}^{N}\frac{\partial}{\partial \mathbf{v}_n}\int_V \begin{pmatrix} \rho_{0q}\varphi - \rho_{0q}\,\mathbf{A}\cdot\mathbf{v} + \\ \rho_0 \vartheta - \rho_0\,\mathbf{U}\cdot\mathbf{v} + \\ \rho_0 \wp - \rho_0\,\mathbf{\Pi}\cdot\mathbf{v} \end{pmatrix} dV_0 =$$

$$= \sum_{n=1}^{N}\frac{\partial}{\partial \mathbf{v}_n}\int_{V_n} \begin{pmatrix} \rho_{0q}\varphi - \rho_{0q}\,\mathbf{A}\cdot\mathbf{v} + \\ \rho_0 \vartheta - \rho_0\,\mathbf{U}\cdot\mathbf{v} + \\ \rho_0 \wp - \rho_0\,\mathbf{\Pi}\cdot\mathbf{v} \end{pmatrix} dV_0 =$$

$$= \sum_{n=1}^{N}\int_{V_n}\frac{\partial}{\partial \mathbf{v}_n} \begin{pmatrix} \rho_{0q}\varphi - \rho_{0q}\,\mathbf{A}\cdot\mathbf{v} + \\ \rho_0 \vartheta - \rho_0\,\mathbf{U}\cdot\mathbf{v} + \\ \rho_0 \wp - \rho_0\,\mathbf{\Pi}\cdot\mathbf{v} \end{pmatrix} dV_0 =$$

$$= \frac{1}{c}\int_V \frac{\partial}{\partial \mathbf{v}} \begin{pmatrix} \rho_{0q}\varphi - \rho_{0q}\,\mathbf{A}\cdot\mathbf{v} + \\ \rho_0 \vartheta - \rho_0\,\mathbf{U}\cdot\mathbf{v} + \\ \rho_0 \wp - \rho_0\,\mathbf{\Pi}\cdot\mathbf{v} \end{pmatrix} u^0 \sqrt{-g}\, dx^1 dx^2 dx^3. \qquad \text{(B6)}$$

In (B6), it was taken into account that when taking the partial derivative $\frac{\partial}{\partial \mathbf{v}_n}$ with respect to the velocity $\mathbf{v}_n$ of a particle with number $n$, the integral $\int_V \begin{pmatrix} \rho_{0q}\varphi - \rho_{0q}\,\mathbf{A}\cdot\mathbf{v} + \rho_0\vartheta - \\ \rho_0\,\mathbf{U}\cdot\mathbf{v} + \rho_0\wp - \rho_0\,\mathbf{\Pi}\cdot\mathbf{v} \end{pmatrix} dV_0$ over the volume of matter can be replaced by the integral $\int_{V_n} \begin{pmatrix} \rho_{0q}\varphi - \rho_{0q}\,\mathbf{A}\cdot\mathbf{v} + \rho_0\vartheta - \\ \rho_0\,\mathbf{U}\cdot\mathbf{v} + \rho_0\wp - \rho_0\,\mathbf{\Pi}\cdot\mathbf{v} \end{pmatrix} dV_0$ over the volume of this one particle with number $n$.

Substituting (B6) into (B3) gives the following:

$$\mathbf{P}'_i = \sum_{n=1}^{N}\frac{\partial}{\partial \mathbf{v}_n}\int_V \begin{pmatrix} -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} \\ -\frac{c^2}{16\pi\eta}u_{\mu\nu}u^{\mu\nu} \\ -\frac{c^2}{16\pi\sigma}f_{\mu\nu}f^{\mu\nu} \\ +\frac{1}{2}c\,k\,R \end{pmatrix} \sqrt{-g}\, dx^1 dx^2 dx^3. \qquad \text{(B7)}$$

Equating momentum (B7) to momentum (B1) for vector fields, we obtain another expression in which the scalar curvature $R$ inside the body in GTR$^m$ is expressed in terms of other quantities:

$$\mathbf{P}'_i = \mathbf{P}_i. \qquad \text{(B8)}$$

Outside matter formula (B1) for vector fields remains valid and gives the momentum of field associated with the matter and commoving with it. In this case, in (B1), the first integral vanishes because the mass density $\rho_0$ and the charge density $\rho_{0q}$ outside matter are equal to zero. In addition, the tensor invariants associated with acceleration field and pressure field are equal to zero. As a result, in (B1) only the sum remains for all those particles that generate electromagnetic and gravitational fields

$$\mathbf{P}_o = \sum_{n=1}^{N}\frac{\partial}{\partial \mathbf{v}_n}\int_V \begin{pmatrix} -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} \\ +\frac{c^2}{16\pi G}\Phi_{\mu\nu}\Phi^{\mu\nu} \end{pmatrix} \sqrt{-g}\, dx^1 dx^2 dx^3. \qquad \text{(B9)}$$

Similarly, from (B2) taking into account the relation $R = 4\Lambda$, for the field momentum outside matter in GTR$^m$ we find

$$\mathbf{P}'_o = \sum_{n=1}^{N}\frac{\partial}{\partial \mathbf{v}_n}\int_V \begin{pmatrix} -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} \\ +\frac{1}{2}c\,k\,R \end{pmatrix} \sqrt{-g}\, dx^1 dx^2 dx^3. \qquad \text{(B10)}$$

The equality of momenta (B9) and (B10) gives a relation that allows us to estimate the value of scalar curvature $R$ in GTR$^m$ outside matter

$$\mathbf{P}'_0 = \mathbf{P}_o. \qquad \text{(B11)}$$

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