Time-Dependent Variational Principle Treatment of Quantum Phase Transition

Azar Vafafard[1,2,*], Alireza Nourmandipour[3], and Roberto Franzosi[4]
1. Faculty of Physics, University of Tabriz, Tabriz, Iran
2. Quantum Technology Center, University of Tabriz, Iran
3. Department of Physics, Sirjan University of Technology, Sirjan, Iran
4. DSFTA, University of Siena, Siena, Italy

E-mail: vafafard@tabrizu.ac.ir

Abstract
Phase transitions are fundamental phenomena in physics, characterized by abrupt changes in the properties of a system. While classical phase transitions occur due to thermal fluctuations, quantum phase transitions (QPTs) are driven by quantum fluctuations at zero temperature. In this work, we explore the presence of QPTs in cavity quantum electrodynamics systems using the Time-Dependent Variational Principle (TDVP), a semi-classical approach for analyzing complex quantum systems. Beginning with the Rabi model, where a single qubit interacts with a single-mode cavity field, we examine the influence of counter-rotating terms on the system's ground state properties. Subsequently, we extend our analysis to the Jaynes-Cummings model, where rotating-wave approximation applies, and finally, to the Dicke model, which considers the collective interaction of multiple qubits with a bosonic mode. For each model, we derive analytical expressions for the ground state properties and identify critical coupling strengths indicative of phase transitions. Our findings reveal second-order quantum phase transitions, including superradiant phases with distinct ground state behaviors, emphasizing the utility of TDVP in understanding QPTs across a variety of systems.

Keywords: time-dependent variation principle, Hamiltonian, quantum phase transition, qubit

1. Introduction

A phase transition typically refers to a fundamental change in the state of a system, characterized by an abrupt shift in one of its parameters, known as the **order parameter** [1]. Familiar examples include the melting of ice, the loss of ferromagnetism, and superfluid-Mott insulator phase transitions in optical lattices [2]. Other phenomena, such as liquid crystals [3], Bose-Einstein condensates [4], and superconductivity [5], also exemplify such transitions. Classical phase transitions are driven primarily by thermal fluctuations and are associated with singularities in thermodynamic quantities [6]. These transitions are marked by macroscopic order, such as crystal structures or magnetization [7]. The order of the transition is determined by the lowest derivative of the free energy that exhibits a discontinuity. Based on this criterion, phase transitions are classified into two categories: first-order transitions, where the first derivative of the free energy is discontinuous, and second-order (or continuous) transitions, where the first derivatives are continuous, but the second derivatives are discontinuous [2].

While classical phase transitions typically occur at finite temperatures, QPTs occur at absolute zero and are driven by quantum fluctuations. These transitions result in ordering of the system's ground-state properties and are often accompanied by spontaneous symmetry breaking [8]. QPTs play a vital role in understanding fundamental phenomena such as phases of matter, mass generation in high-energy physics, magnetism, and superconductivity [9]. Moreover, they have practical significance in condensed matter systems, including magnetic insulators, heavy-fermion compounds, high-temperature superconductors, and two-dimensional electron gases [9]. Theoretically [10] and experimentally [11], QPTs have been shown to typically occur in the thermodynamic limit, where the number of two-level atoms in the system becomes very large. This phenomenon is often described using models like the Dicke model [12] or the Lipkin-Meshkov-Glick model [13]. However, it has also been theoretically demonstrated that QPTs can occur in systems involving just a single atom interacting with a cavity field. These include the Jaynes-Cummings model (under the rotating wave approximation) [9] and the Rabi model (without the rotating wave approximation) [14]. Recent experiments have confirmed the occurrence of QPTs in the Rabi model using a trapped ion in a Paul trap [15], where the spin-up state population and the average phonon number of the ion were measured as order parameters.

From a computational standpoint, a fully quantum mechanical treatment of QPTs is often complex and mathematically intensive [9,14]. Despite the advantages of this approach—such as probing superradiant QPT dynamics [16], excited-state quantum phase transitions [17], and dissipative phase transitions [18]—a semi-classical approach can provide valuable insights, especially for more intricate systems like the quantum Dicke model, where multiple qubits interact with a cavity field. In this context, the TDVP [19] has emerged as a powerful method to describe quantum evolution. TDVP has been successfully applied in various domains, including open quantum systems [20] and quantum neural networks [21].

In this paper, we utilize TDVP to investigate QPTs by analyzing the ground-state energy of several models. We begin with the Rabi model, where non-conservative terms are retained. Next, we study the Jaynes-Cummings model. Finally, we extend our approach to the Dicke model, involving an arbitrary number of qubits interacting with a single cavity mode. Our results reveal evidence of phase transitions in the ground states of these models.

The remainder of the paper is organized as follows. Section 2 investigates the Rabi model. Section 3 addresses the Jaynes-Cummings model. In Section 4, we apply the proposed method to the Dicke model. Finally, Section 5 concludes the paper with a summary of our findings.

2. Rabi Model

Consider a system in which a qubit with transition frequency ω_z is interacting with a single-mode of a high-Q cavity in the Rabi model. The Hamiltonian describing the whole system is written as $(\hbar = 1)$:

$$\hat{H}_{\text{Rabi}} = \frac{\omega_z}{2}\hat{\sigma}_z + \omega_c \hat{a}^\dagger \hat{a} + \lambda(\hat{a} + \hat{a}^\dagger)\hat{\sigma}_x, \tag{1}$$

where $\hat{\sigma}_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$ is the population inversion operator of the qubit with transition frequency ω_z. ω_c represents the frequency of the cavity quantized mode. $\hat{\sigma}_x = \hat{\sigma}_+ + \hat{\sigma}_-$ with $\hat{\sigma}_+ = |\uparrow\rangle\langle\downarrow|$ ($\hat{\sigma}_- = |\downarrow\rangle\langle\uparrow|$) denotes the raising (lowering) operator for the qubit, while $\hat{a}$ ($\hat{a}^\dagger$) is the annihilation (creation) operators of the cavity mode. Finally, λ represents the coupling strength of the interaction of the qubit with the cavity mode. Although the total number of excitations $\hat{N}_{\text{tot}} = \hat{a}^\dagger \hat{a} + \hat{\sigma}_+ \hat{\sigma}_-$ is not conserved in quantum Rabi model due to the presence of the counter-rotating terms, there is a Z_2 symmetry by which the parity operator

$$\hat{\Pi} = e^{i\pi\left[\hat{a}^\dagger \hat{a} + (1/2)(\hat{I} + \hat{\sigma}_z)\right]} \tag{2}$$

commutes with the Hamiltonian, i.e., $\left[\hat{\Pi}, \hat{H}_{\text{Rabi}}\right] = 0$, and therefore it is conserved. This corresponds to the simultaneous replacement of $\hat{a} \mapsto -\hat{a}$ and $\hat{\sigma}_x \mapsto -\hat{\sigma}_x$. Here, we intend to investigate the presence of a quantum phase transition based on a semi-classical method, TDVP. We investigate the possibility of phase transition by studying the ground state of the Rabi Hamiltonian. To apply the Time-Dependent Variational Principle (TDVP), we use a factorized ansatz for the wavefunction. This ansatz assumes a product of a coherent state for the bosonic mode and a variational spinor state for the qubit. The form is inspired by the semiclassical nature of TDVP and is particularly suited to capturing key features of the ground-state behavior. While the ansatz appears in similar form in all three models discussed (Rabi, Jaynes-Cummings, and Dicke), it is contextually adapted to the Hamiltonian and system size of each. For this reason, we relabel the ansatz as Eqs. (3), (19), and (31) for clarity and convenience, even though their mathematical structure is nearly identical. Accordingly, we consider an ansatz

$$|\psi(t)\rangle = e^{iS(t)}|\alpha(t)\rangle|\chi(t)\rangle, \tag{3}$$

in which, $|\alpha(t)\rangle = e^{-\frac{|\alpha(t)|^2}{2}} \sum_{n=0} \frac{\alpha(t)^n (\hat{a}^\dagger)^n}{n!}|0\rangle$ is the coherent state of the cavity field, $|\chi(t)\rangle = \cos[\theta(t)/2]|\uparrow\rangle + \sin[\theta(t)/2]e^{i\phi(t)}|\downarrow\rangle$ is the normalized spin state and $S(t)$ is a phase to be determined. In this relation $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$. Then, using time-dependent Schrödinger equation, we have

$$i\langle\psi(t)|\frac{d}{dt}|\psi(t)\rangle = \langle\psi(t)|\hat{H}_{\text{Rabi}}|\psi(t)\rangle. \tag{4}$$

After some straightforward calculations we obtain

$$\dot{S}(t) = \int_0^t L(t')\mathrm{d}t' \tag{5}$$

in which $L(t)$ is the (semi-classical) Lagrangian associated to the system which takes the form

$$L(t) = i\langle\alpha(t)|\langle\chi(t)|\frac{d}{dt}|\alpha(t)\rangle|\chi(t)\rangle - H_{\text{Rabi}}(\alpha, \alpha^*, \theta, \phi). \tag{6}$$

Here $H_{\text{Rabi}}(\alpha, \alpha^*, \theta, \phi) = \langle\alpha(t)|\langle\chi(t)|\hat{H}_{\text{Rabi}}|\alpha(t)\rangle|\chi(t)\rangle$ is the expectation value of the Hamiltonian with respect to the state (3). The set of variables $\{\alpha, \alpha^*, \theta, \phi\}$ introduces the trajectories $\alpha(t)$, $\alpha^*(t)$, $\theta(t)$ and $\phi(t)$ [19].

This expression for the Lagrangian arises from the application of the Time-Dependent Variational Principle (TDVP), which states that the best approximation to the quantum dynamics within a chosen variational manifold is obtained by minimizing the action $S = \int dt\left(\langle\psi(t)|i\frac{d}{dt}|\psi(t)\rangle - \langle\psi(t)|\hat{H}|\psi(t)\rangle\right)$.

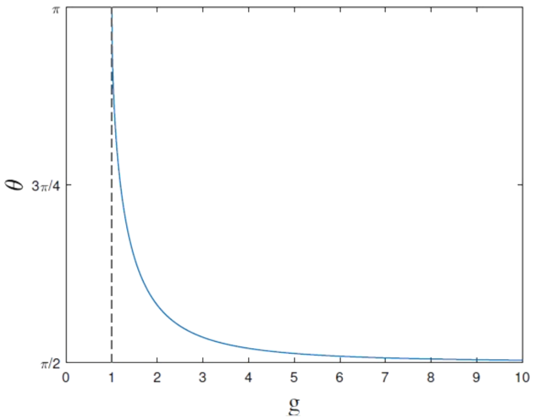

Figure 1: The solution for θ as a function of g. For $g<1$, there is no real solution for θ.

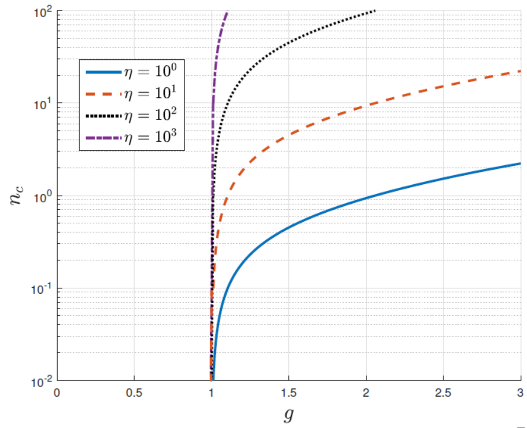

Figure 2: The cavity photon number of the ground state of Rabi model.

Substituting the variational ansatz into this expression yields the effective Lagrangian

$$L(t)=i\langle\psi(t)|\frac{d}{dt}|\psi(t)\rangle-\langle\psi(t)|\hat{H}|\psi(t)\rangle,$$

which governs the dynamics of the variational parameters. This semi-classical treatment provides a tractable approximation to the full quantum evolution, while still capturing key features such as ground-state transitions and critical behavior.

The Lagrangian (6) can be written as a function of these trajectories as follow [19]

$$L(t)=\frac{i}{2}\dot{\alpha}(t)\alpha^*(t)-\frac{i}{2}\dot{\alpha}^*(t)\alpha(t)-\dot{\phi}(t)\sin^2(\theta(t)/2) \tag{7}$$
$$-H_{\text{Rabi}}(\alpha(t),\alpha^*(t),\theta(t),\phi(t)).$$

According to the principle of least action, the equations of motion for the actual trajectories are obtained from Lagrange equations of motions ($\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_i}=0$, $q_i\in\{\alpha,\alpha^*,\theta,\phi\}$) as follow

$$\dot{\alpha}=-i(\alpha\omega_c+\lambda\cos\phi\sin\theta), \tag{8}$$

$$\dot{\alpha}^*=i(\alpha^*\omega_c+\lambda\cos\phi\sin\theta), \tag{9}$$

$$\dot{\theta}=-2\lambda\sin\phi(\alpha+\alpha^*), \tag{10}$$

$$\dot{\phi}=\omega_z-2\lambda\cos\phi(\alpha+\alpha^*)\cot\theta. \tag{11}$$

We are interested in the stationary solution of the above equations of motion. From Eqs. (8) and (9) we obtain

$$\alpha=\alpha^*=-\frac{\lambda\cos\phi\sin\theta}{\omega_c}. \tag{12}$$

Then, Eq. (10) at the stationary state leads to the following equation

$$\frac{2\lambda^2\sin\theta\sin(2\phi)}{\omega_c}=0, \tag{13}$$

which means that $\sin(2\phi)=0$ or $\sin\theta=0$. Let us consider the first case, i.e, $\sin(2\phi)=0$ which gives rise to $\phi=m\pi/2$. Finally, from Eq. (11), one can easily observe that for $\phi=(2n+1)\pi/2$, Eq. (11) leads to $\omega_z=0$ which is not a physical result. On the other hand, for values $\phi=n\pi$ an analytical expression for θ at the stationary state may be obtained as follow

$$\eta\left(1+g^2\cos\theta\right)=0, \tag{14}$$

in which we have used the dimensionless coupling strength $g=2\lambda/\sqrt{\omega_c\omega_z}$ where η is the frequency ration as $\eta=\omega_z/\omega_c$ [14]. The introduction of g allows us to rescale the Hamiltonian in a way that reveals universal features of the quantum phase transition, independent of absolute frequency values. This form also ensures the critical point occurs at $g=1$, simplifying the analysis of ground state behavior and phase boundaries. It is evident that regardless of η, Eq. (14) has no real solution for $g<1$, however, for $g\geq 1$ the analytical solution is

$$\theta=\cos^{-1}(-1/g^2). \tag{15}$$

This indicates a quantum phase transition locating at $g=1$. According to the above solution, $\pi/2\leq\theta\leq\pi$, in which, the upper bound for θ is obtained at $g=1$ which corresponds to the spin state $|\chi\rangle=|\downarrow\rangle$ up to a global phase (see Fig. 1). On the other hand, for large values of g, $\theta\to\pi/2$. This leads a superposition of the ground and the excited state for the qubits, i.e., $|\chi\rangle=\frac{1}{\sqrt{2}}\left(|\uparrow\rangle+(-1)^n|\downarrow\rangle\right)$.

In this scenario, the number of photons in the cavity becomes proportional to $\eta=\omega_z/\omega_c$. This means a *superradiance* occurrence in the number of the photons in the cavity in the ground state of the system. Explicitly, the cavity photon number $n_c=|\alpha|^2$ takes the following analytical expression

$$n_c=\frac{\eta}{4}\left(g^2-\frac{1}{g^2}\right) \tag{16}$$

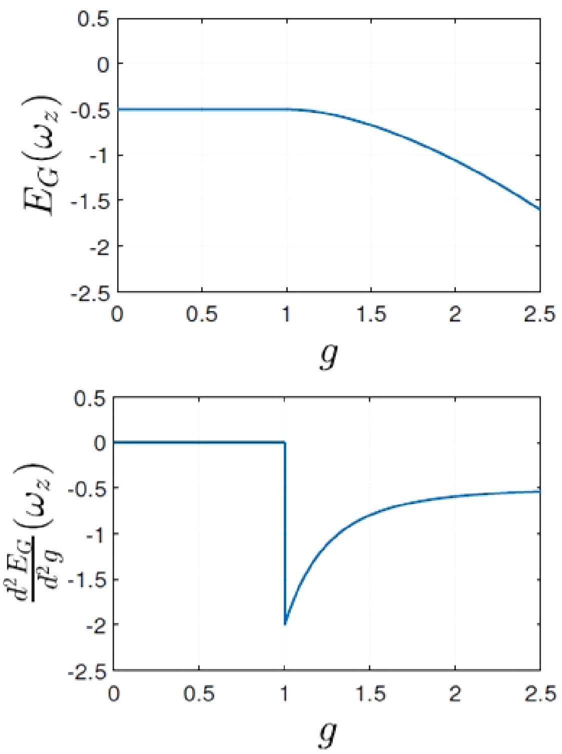

Figure 3: The ground state energy of the Rabi model i.e., E_G (top plot) and d^2E_G/dg^2 (bottom plot) versus g.

which is greater than zero for $g \geq 1$. Fig. 2 illustrates the behaviour of n_c as a function of g for several values of η. It is evident the quantum phase transition at the critical value $g = g_c = 1$.

It is also interesting to investigate the ground state energy based on the presented method. The ground state energy of the system may be obtained via $E_G(g) = \langle\psi|\hat{H}_{\text{Rabi}}|\psi\rangle$ which takes the following analytical expression for $g \geq 1$

$$E_G(g) = -\frac{\omega_z}{4}\left(g^2 + g^{-2}\right), \ (g \geq 1). \tag{17}$$

We should point out that the above relation is valid only for $g \geq 1$, i.e., the superradiant phase. In order to find the relevant expressions for $g \leq 1$, i.e., normal phase, we observe that the other solution for Eq. (13) is $\sin\theta = 0$ which gives rise to $\theta = 0$ and π. For both values of θ, we have $n_c = 0$. However, $\theta = \pi$ is the correct answer. This is because for $\theta = \pi$ the ground state of the system is obtained, i.e., $|0\rangle|\downarrow\rangle$. Then the ground state energy of the system is obtained as $E_G = -\omega_z/2$ for $g < 1$ which indicates the normal phase. According to Fig. 3, while E_G (as well as its first derivative with respect to g) is continues, there exists a discontinuity in $\frac{d^2E_G}{dg^2}$ at $g = 1$ indicating a second-order quantum phase transition.

3. Jaynes-Cummings Model

We now turn into a model in which the qubit is interacting with a single-mode of a high-Q cavity in the Jaynes-Cummings model via the following Hamiltonian ($\hbar = 1$):

$$\hat{H}_{\text{JC}} = \frac{\omega_z}{2}\hat{\sigma}_z + \omega_c\hat{a}^\dagger\hat{a} + \lambda(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-), \tag{18}$$

On the contrary to Rabi model, the conserved total number of excitations $\hat{N}_{\text{tot}} = \hat{a}^\dagger\hat{a} + \hat{\sigma}_+\hat{\sigma}_-$ implies the $U(1)$-continuous symmetry, according to which, only the transitions $|n+1,\downarrow\rangle \leftrightarrow |n,\uparrow\rangle$ (or $|n,\downarrow\rangle \leftrightarrow |n-1,\uparrow\rangle$) are allowed. However, we emphasis that in the view of occurring QPT we allow the system to break the $U(1)$-continuous symmetry, i.e., transitions $|n,\downarrow\rangle \leftrightarrow |n,\uparrow\rangle$ and etc. are also allowed.

Therefore, we should expect that the conservation of total number of excitations is not hold any more. We adopt the same variational ansatz as in the previous section, with the parameters now adapted to the Jaynes-Cummings Hamiltonian:

$$|\psi(t)\rangle = e^{iS(t)}|\alpha(t)\rangle|\chi(t)\rangle, \tag{19}$$

in which, $|\alpha(t)\rangle = e^{-\frac{|\alpha(t)|^2}{2}}\sum_{n=0}\frac{\alpha(t)^n(a^\dagger)^n}{n!}|0\rangle$ is the coherent state of the cavity field, $|\chi(t)\rangle = \cos[\theta(t)/2]|\uparrow\rangle + \sin[\theta(t)/2]e^{i\phi(t)}|\downarrow\rangle$ is the normalized spin state and $S(t)$ is a phase to be determined. Again, using time-dependent Schrödinger equation, we have

$$i\langle\psi(t)|\frac{d}{dt}|\psi(t)\rangle = \langle\psi(t)|\hat{H}_{\text{JC}}|\psi(t)\rangle. \tag{20}$$

The same procedure will lead to similar equations of motion for relevant parameters, however in this case

$$L(t) = i\langle\alpha(t)|\langle\chi(t)|\frac{d}{dt}|\alpha(t)\rangle|\chi(t)\rangle - H_{\text{JC}}(\alpha,\alpha^*,\theta,\phi), \tag{21}$$

where, $H_{\text{JC}}(\alpha,\alpha^*,\theta,\phi) = \langle\alpha(t)|\langle\chi(t)|\hat{H}_{\text{JC}}|\alpha(t)\rangle|\chi(t)\rangle$ is the expectation value of the Hamiltonian with respect to the state (19). According to the principle of least action, the equations of motion for the actual trajectories are obtained from Lagrange equations of motions ($\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$, in which $q_i \in \{\alpha,\alpha^*,\theta,\phi\}$) as follow

$$\dot{\alpha} = -\frac{1}{2}i\left(2\alpha\omega_c + \lambda e^{-i\phi}\sin\theta\right), \tag{22}$$

$$\dot{\alpha}^* = \frac{1}{2}i\left(2\alpha^*\omega_c + \lambda e^{i\phi}\sin\theta\right), \tag{23}$$

$$\dot{\theta} = i\lambda\left(e^{i\phi}\alpha - e^{-i\phi}\alpha^*\right), \tag{24}$$

$$\dot{\phi} = \omega_z - \lambda\left(e^{i\phi}\alpha + e^{-i\phi}\alpha^*\right)\cot\theta. \tag{25}$$

We are interested in the stationary solution of the above equations of motion. From Eqs. (22) and (23), we obtain

$$\alpha = (\alpha^*)^* = -\frac{e^{-i\phi}\lambda\sin\theta}{2\omega_c}. \tag{26}$$

Finally, from Eq. (25), the equation of motion for θ as the stationary state is obtained

$$\omega_z\left(1 + g'^2\cos\theta\right) = 0, \tag{27}$$

in which we have used the dimensionless coupling strength $g' = \lambda/\sqrt{\omega_c\omega_z}$ [9]. We also introduce the frequency ratio $\eta = \omega_z/\omega_c$ which provides a normalized measure of the interaction strength relative to the qubit frequency. As in the Rabi model, this rescaling simplifies the analytical treatment and highlights the universal nature of the critical behavior. Although the Jaynes-Cummings model is integrable and has different symmetry properties, we find that a critical value $g' = 1$ also marks the onset of quantum criticality in the semiclassical treatment. It is evident that regardless of η, Eq. (27) has no real solution for $g' < 1$, however, for $g' \geq 1$ the analytical solution is

$$\theta = \cos^{-1}(-1/g'^2). \tag{28}$$

This indicates a quantum phase transition locating at $g' = 1$. According to the above solution, $\pi/2 \leq \theta \leq \pi$, in which, the upper bound for θ is obtained at $g' = 1$ which corresponds to the spin state $|\chi\rangle = |\downarrow\rangle$ up to a global phase. In order to investigate the presence of quantum phase transition, we should point out that with the optimal value for θ, we have the following expression for α:

$$\begin{aligned} \alpha &= -e^{-i\phi}\,\frac{\sqrt{\eta(g^2 - g^{-2})}}{4}, \\ \alpha^* &= -e^{i\phi}\,\frac{\sqrt{\eta(g^2 - g^{-2})}}{4}. \end{aligned} \tag{29}$$

By comparing the above two equations, we observe that the factor $\sqrt{\eta(g^2 - g^{-2})}$ must be real. This means that for $g < 1$, $\alpha = 0$. Then, according to Eq. (26), $\sin\theta = 0$ for $g < 1$. Similar to the Rabi model, only $\theta = \pi$ leads to the correct ground state of the system in normal phase $g < 1$.

4. Dicke Model

Now we are in the position to study a more general model, i.e., Dicke model which describes a single bosonic mode interacting collectively with a set of N qubits via the following Hamiltonian ($\hbar = 1$):

$$\hat{H}_{\text{Dicke}} = \omega_z J_z + \omega_c a^\dagger a + \frac{\lambda}{\sqrt{N}}(a + a^\dagger)J_x, \tag{30}$$

in which $J_\alpha = \sum_{\mu=1}^{N}\frac{\sigma_\alpha^\mu}{2}$, with $\alpha = 1,2,3$. As in the previous models, we adopt a time-dependent variational ansatz combining a bosonic coherent state with a spin-coherent state that captures the collective behavior of the N qubits. While structurally similar to the earlier ansatz forms, this version is tailored to the Dicke Hamiltonian and incorporates collective spin degrees of freedom. Therefore, we adopt the following ansatz

$$|\psi(t)\rangle = e^{iS(t)}|\alpha(t)\rangle|\theta(t),\phi(t)\rangle, \tag{31}$$

in which $|\theta,\phi\rangle$ is the normalized spin coherent state which is defined as

$$|\theta,\phi\rangle = \left(\frac{1}{1+|\tau|^2}\right)^J e^{\tau\hat{J}_+}|J,-J\rangle, \tag{32}$$

where $|J,-J\rangle = \bigotimes_{l=1}^{N}|\downarrow\rangle_l$ is the eigenstate of $\hat{J}_z$ with eigenvalue $-J$ with $J = N/2$. Also

$$\tau = e^{-i\phi}\tan\left(\frac{\theta}{2}\right). \tag{33}$$

Again the time-dependent Schrödinger equation leads us to the similar equation of motion with

$$\begin{aligned} L(t) = {} & i\langle\alpha(t)|\langle\chi(t)|\frac{d}{dt}|\alpha(t)\rangle|\chi(t)\rangle \\ & - H_{\text{Dicke}}(\alpha,\alpha^*,\theta,\phi), \end{aligned} \tag{34}$$

where,
$H_{\text{Dicke}}(\alpha,\alpha^*,\theta,\phi) = \langle\alpha(t)|\langle\chi(t)|\hat{H}_{\text{Dicke}}|\alpha(t)\rangle|\chi(t)\rangle$.
According to the principle of least action, the equations of motion for the actual trajectories are obtained from Lagrange equations of motions ($\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$, in which $q_i \in \{\alpha,\alpha^*,\theta,\phi\}$) as follow

$$i\dot{\alpha} = \omega_c\alpha + \frac{\lambda}{\sqrt{N}}J\sin\theta\cos\phi, \tag{35}$$

$$-i\dot{\alpha}^* = \omega_c\alpha^* + \frac{\lambda}{\sqrt{N}}J\sin\theta\cos\phi, \tag{36}$$

$$\dot{\theta} = \frac{\lambda}{\sqrt{N}}\left(\alpha + \alpha^*\right)\sin\phi, \tag{37}$$

$$\dot{\varphi} = \frac{1}{J\sin\theta}\left(\omega_z J\sin\theta - \frac{\lambda}{\sqrt{N}}J\left(\alpha + \alpha^*\right)\cos\theta\cos\varphi\right). \tag{38}$$

Then, at the stationary state, from Eq. (37) we obtain $\phi = k\pi$ with $k \in \mathbb{Z}$. Then Eqs. (35) and (36) lead to

$$\alpha = \alpha^* = -\frac{\lambda J}{\sqrt{N}\omega_c}\sin\theta(-1)^k. \tag{39}$$

Finally, from Eq. (38), we obtain the following equation of motion for θ as the stationary state

$$\left(1 + g'^2\cos\theta\right)J\sin\theta = 0, \tag{40}$$

in which $g' = \lambda/\sqrt{\omega_c \omega_z}$ [9]. We also introduce the frequency ratio $\eta = \omega_z/\omega_c$. Again, it is evident that regardless of η, Eq. (40) has no real solution for $g' < 1$, however, for $g' \geq 1$ the analytical solution is

$$\theta = \cos^{-1}(-1/g'^{2}). \tag{41}$$

This indicates a quantum phase transition locating at $g' = 1$. According to the above solution, $\pi/2 \leq \theta \leq \pi$, in which, the upper bound for θ is obtained at $g' = 1$ which corresponds to the spin coherent state $|J, -J\rangle$ up to a global phase. In order to investigate the presence of quantum phase transition, we should point out that with the optimal value for θ, we have the following expression for α:

$$\alpha = \alpha^* = -J\sqrt{\frac{\eta}{N}(g'^{2} - g'^{-2})}(-1)^k, \tag{42}$$

By comparing the above two equations, we observe that the factor $\sqrt{\frac{\eta}{N}(g'^{2} - g'^{-2})}$ must be real. This means that for $g' < 1$, $\alpha = 0$. Then, according to Eq. (39), $\sin\theta = 0$ for $g' < 1$. Similar to the Rabi model, only $\theta = \pi$ leads to the correct ground state of the system in normal phase $g' < 1$.

The semi-classical approach based on the time-dependent variational principle successfully captures the quantum phase transition in the Dicke model. The results are consistent with the expected superradiant behavior and demonstrate the critical role of collective interactions in driving the phase transition. This framework provides a foundation for exploring more complex models and their experimental realizations.

5. Conclusion

In this study, we utilized the Time-Dependent Variational Principle to explore quantum phase transitions in three cornerstone models of cavity quantum electrodynamics: the Rabi model, the Jaynes-Cummings model, and the Dicke model. By leveraging a semi-classical approach, we derived effective equations of motion and analyzed the stationary solutions to uncover critical behaviours in the ground state properties of these systems. Our findings shed light on the mechanisms underlying QPTs, offering both theoretical insights and practical implications for quantum technologies.

For the Rabi model, we identified a critical coupling strength where a second-order quantum phase transition occurs. The inclusion of counter-rotating terms in the Hamiltonian led to the emergence of a superradiant phase characterized by macroscopic photon occupation in the cavity. This highlights the profound influence of counter-rotating terms on the system's dynamics and their role in facilitating phase transitions.

In the Jaynes-Cummings model, which incorporates the rotating-wave approximation, the analysis revealed similar quantum-critical phenomena. The study of this simplified model not only reinforced the results obtained for the Rabi model but also emphasized the effects of simplifying assumptions on the quantum phase transition behaviour.

The Dicke model offered a more complex and general scenario, with multiple qubits collectively interacting with the bosonic mode. Our results demonstrated that the critical coupling strength decreases as the number of qubits increases, signifying the enhanced role of collective interactions in driving superradiant phases. We derived analytical expressions for the ground state properties, identifying a second-order quantum phase transition at a critical point where the system transitions from a normal to a superradiant phase. This provides valuable insights into the scalability of QPTs in many-body systems.

Additionally, our analysis underscores the role of the light-matter coupling strength as a key control parameter in experimental platforms, enabling the observation and manipulation of quantum phase transitions in cavity QED systems.

The significance of these findings extends beyond the theoretical framework, as QPTs in light-matter systems have broad implications for the development of quantum technologies. Superradiant phases, for instance, hold promise for enhancing light-matter coupling in quantum information processing, quantum sensing, and quantum simulation. The use of TDVP in our study offers a robust and computationally efficient approach for exploring such complex phenomena, bridging the gap between exact quantum methods and practical semi-classical approximations.

Looking forward, our work sets the stage for investigating more sophisticated models, such as systems with non-Markovian environments, dissipative phase transitions, or multimode bosonic fields. Moreover, experimental realizations of these transitions, particularly in platforms such as trapped ions, superconducting qubits, and ultracold atomic gases, could provide direct validation of our theoretical predictions. By expanding our understanding of QPTs and their associated critical phenomena, this study contributes to advancing both fundamental science and the frontier of quantum technology applications.

References

1. H Nishimori and G Ortiz, "Elements of Phase Transitions and Critical Phenomena", Oxford University Press, (2010).
2. A Carollo, D Valenti, and B Spagnolo, Physi. Rep. 838 (2020) 1.
3. KA Takeuchi, M Kuroda, H Chat´e, and M Sano, Phys. Rev. E 80 (2009) 051116.
4. E Michon, et al., New J. Phys. 20 (2018).

5. A Pippard, Physica 22 (1956) 99.
6. N Goldenfeld, "Lectures On Phase Transitions And The Renormalization Group", Frontiers in Physics, Westview Press, (1992)
7. HE Stanley, "Phase transitions and critical phenomena", Clarendon Press, Oxford, (1971).
8. M Vojta, Rep. Prog. Phys. 66 (2003) 2069.
9. MJ Hwang and MB Plenio, Phys. Rev. Lett. 117 (2016) 123602.
10. N Lambert, C Emary, and T Brandes, Phys. Rev. Lett. 92 (2004) 073602.
11. B Dietz, et al., Phys. Rev. B 88 (2013) 104101.
12. C Emary and T Brandes, Phys. Rev. E 67 (2003) 066203.
13. H Lipkin, N Meshkov, and A Glick, Nucl. Phys. 62 (1965) 188.
14. MJ Hwang, R Puebla, and MB Plenio, Phys. Rev. Lett. 115 (2015) 180404.
15. ML Cai, et al., Nat. Commun. 12 (2021) 1126.
16. R Puebla, et al., Phys. Rev. Lett. 118 (2017) 073001.
17. R Puebla, MJ Hwang, and MB Plenio, Phys. Rev. A 94 (2016) 023835.
18. MJ Hwang, P Rabl, and MB. Plenio, Phys. Rev. A 97 (2018) 013825.
19. P Kramer, J. Phys. Con. Ser. 99 (2008) 012009.
20. CV Kraus and TJ Osborne, Phys. Rev. A 86 (2012) 062115 (2012).
21. M Reh, M Schmitt, and M Garttner, Phys. Rev. Lett. 127, (2021) 230501.