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Soliton behaviour in models of baroclinic instability

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Abstract
Here we concern ourselves with the derivation of a system of evolution equations for slowly varying amplitude of a baroclinic wave packet. The self-induced transparency, sine-Gordon, and nonlinear Schrodinger equations, all of which possess soliton solutions, each arise for different inviscid limits. The presence of viscosity, however, alters the form of the evolution equations and changes the character of the solutions from highly predictable soliton solutions to unpredictable chaotic solutions. When viscosity is weak, equations related to the Lorenz attractor equations obtain, while for strong viscosity the Ginzburg-Landau equation obtain.
Keywords: sine-Gordon equation, soliton, baroclinic instability

1. Introduction

One of the major concerns of the meteordogist is the degree of predictability of atmospheric motions. The classic remarks made by Lorenz [1], in the now-celebrated paper in which deterministic equations were first shown to exhibit aperiodic and consequently predictable behaviour, that it may be impossible to predict the weather accurately beyond a few days, only too truly reflect the current state of affairs.

We shall not be concerned, in this article, with direct modeling of atmospheric predictability. Instead we shall concentrate on the phenomena occurring in simple models exposing the essential physical behaviour, and we shall demonstrate that under certain conditions, coherent persistent behaviour is possible.

Cyclones, anticyclones and their associated frontal system are a prominent feature of the mid-latitude, westerlies of the Earth's lower atmosphere. Their importance as weather bearing systems and more generally their role in the general circulation of the atmosphere is well-known if not yet well understood. Their occurrence and rigidly changing behaviour is strongly influenced by the existence of large scale "Long waves", which are remarkable for their persistence and coherence over longer periods of time. Both phenomena owe their existence to the availability of potential energy associated with the baroclinicity of the fluid, i. e. the non-coincidence of surfaces of constant gravitational potential and constant density, which is a possible equilibrium in a rotating system. Such an equilibrium is unstable and wave-like perturbations can develop at the expense of the potential energy if the trajectories of fluid particles are contained within the geopotentials and isopynals. This process is known as sloping convection or baroclinic instability and the consequent waves as baroclinic waves. Mathematical models are almost invariably infinite channel models; the simplest are the heterogeneous model due to Eady [2], and the two-layer model due to Phillips [3]. The basic state for the Eady model is one of linear vertical shear; for the two-layer model the zonal velocity is constant in each layer. Over the past couple of decades a number of authors have studied various aspects of the weakly nonlinear behaviour of wave-like perturbations to the basic states of both models, when the amplitude is permitted to vary slowly in both time and/or space.

The motivation for considering baroclinic wave-packet behaviour lay partly in the experiments of Hide et al. [4] and other experiments who had long observed a motivation of the baroclinic wave and a recurrence property of the data. They interpreted the motivation in terms of a triad interaction between the dominant wave, its side-bands and the long wave. Subsequent numerical integrations performed by Farnell and James [5] led to support this conjecture and a wave-packet model was thought to be an alternative and possibly better way of viewing log-wave modulation. The material of this paper draws heavily upon the main results appearing in Gibbon et. al. [6], Moroz [7] and Moroz and Brindley [8]. Section 2 contains descriptions of both the continuously stratified and the two-layer models as well as the linear stability theory of both. In Section 3 we indicate the nonlinear theory and show how the completely integrable equations arise. Finally in Section 4 there is a comparison of theoretical and experimental results.

2. The Models

The model equations, known as the quasi-geostrophic potential vorticity equations, are obtained from the Navier-Stokes equations, the equation of continuity of mass and other subsidiary relations, under certain assumptions as listed below.

(i) Inviscid incompressible fluid(s).

(ii) Stable density stratification.

(iii) The Boussinesq approximation i.e. density variations are only taken into account in the boundary term in the Navier-Stokes equations.

(iv) Infinite rectangular channel model: infinite in the x-extent and bounded laterally by vertical frictionless walls and above and below the rigid wall.

(v) For the layered model we make the Rossby β-plane approximation and allow horizontal variations of the Coriolis parameter with cross channel coordinate, y, which is one way of modelling the Earth's sphericity in certain coordinates.

(vi) For the continuously stratified model we permit the lid and base of the channel to have a lateral slope, the slope being no longer than the slope of the isotherms.

(vii) The systems rotate rapidly about a vertical axis with angular speed $\mathbf{\Omega}$ so that the dominant balance is between the Coriolis force $2\mathbf{\Omega}\times\mathbf{u}$ and the pressure gradient ∇_p, dimensional analysis shows that acceleration and advection are an order of magnitude smaller than these two forces.

(viii) The basic state satisfies the thermal wind relation: vertical shear horizontal density gradient.

(ix) The motion is nearly two dimensional in the horizontal plane.

The derivation of the quasi-geostrophic equations is straightforward and will not be given here. The details can be found elsewhere [9]; here we merely quote the results. For a continuity stratified model we have to solve

$$\left[\frac{\partial}{\partial t}+J_{(x,y)}(\psi,)\right]\nabla_B^2\psi=0 \tag{1}$$

subject to

$$\psi_x=0 \text{ on } y=\pm\frac{1}{2} \tag{2}$$

$$\lim_{x\to\infty}\frac{1}{2}\int_{-x}^{x}\psi_{y'}dx'=0 \text{ on } y=\pm\frac{1}{2} \tag{3}$$

and

$$\left[\frac{\partial}{\partial t}+J_{(x,y)}(\psi,)\right]\psi_z+S_i\psi_x=0 \text{ on } z=\pm\frac{1}{2} \tag{4}$$

where

$$\nabla_B^2\equiv\frac{\partial^2}{\partial z^2}+B\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

$$J_{x,y}(f,g)=f_x g_y-f_y g_x,$$

and ψ is the non-dimensional pressure field. In the following we assume S_1=-S_2. For a two-layer model, we have

$$\left[\frac{\partial}{\partial t}+J_{(x,y)}(\psi_i,)\right]\left[\nabla^2\psi_i+(-1)^{i+1}F(\psi_1-\psi_2)+\beta y\right] = 0, \quad i=1,2 \qquad (1^*)$$

subject to

$$\psi_{ix}=0 \text{ on } y=0,\ 1 \qquad (2^*)$$

$$\lim_{x\to\infty}\frac{1}{2x}\int_{-x}^{x}\psi_{iy'}dx'=0 \text{ on } y=0,1 \qquad (3^*)$$

Here we have introduced the non-dimensional parameters:

B	Burger Number	(Stratification parameter),
S_i(i=1,2)	Non-dimensional boundary slopes	(Dispersion parameter),
F	Internal rotational Froude number	(Stratification parameter),
β	β-effect	(Dispersion parameter).

We seek wave-like perturbations to a simple zonal flow. Replacing ψ by $-$ yz $+\psi$ for the continuous model

$$\psi=(P\cosh 2qz+Q\sinh 2qz)e^{ik(x-ct)}\sin m\pi\left(y+\frac{1}{2}\right) \qquad (5)$$

and replacing

ψ_i by $-u_iy+\psi_i$ for two layer model.

$$\psi_i=\begin{pmatrix}1\\ \gamma\end{pmatrix}e^{ik(x-ct)}\sin m\pi y \qquad (5^*)$$

and linearize the resulting equations, we obtain characteristic relations:

$$\begin{aligned}H_E &= 4q^2c^2+2S_2q(\tanh q+\coth q)c+S_2^2\\ &\quad-(1+q^2)+q(\tanh q+\coth q)\\ 4q^2 &= B(k^2+m^2\pi^2)\end{aligned} \qquad (6)$$

and

$$\begin{aligned}H_L &= a^2\left(a^2+2F\right)c^2-c\left[a^2\left(a^2+2F\right)(u_1+u_2)\right.\\ &\quad\left.-2\beta\left(a^2+F\right)\right]+a^2\left(a^2+2F\right)u_1u_2+\beta^2\\ &\quad+Fa^2u_c^2-\beta(u_1+u_2)\left(a^2+F\right)\end{aligned} \qquad (6^*)$$

for the two-layer model, where

$$a^2=k^2+m^2\pi^2$$

The dispersion relations are found by solving H=0 for c. The condition for marginal stability kc_i=0 yields surfaces which separate stable and unstable regions for parameter space. These surfaces often have a local form dominated by one particular parameter. For the continuity-stratified model, S_2 is the relevant stability parameter and the condition for marginal stability is

$$S_2^2=\frac{4\left[q(\tanh q+\coth q)-\left(1+q^2\right)\right]}{(\tanh q-\coth q)^2} \qquad (7)$$

for the two layer model it is u_c=u_1-u_2 and

$$u_c^2=\frac{4\beta^2F^2}{a^4\left(4F^2-a^4\right)} \qquad (7^*)$$

The validity of a wave packet analysis rests on the existence of suitable behaviour in the linear problem. For small departures of order Δ of the stability parameter, Δ, from criticability, a band of wave numbers of width $\Delta^{1/2}$ is unstable. Moroz and Brindley [8] have shown that the stationary point in the "Eady" model occurs for S_2=1 and at that point we have a coalescence of two modes with identical phase speeds, (c=-1/2) but different group speeds.

3. Nonlinear theory

The results from linear theory suggest that we introduce new variables X_1 and T_1, scaled respectively on the bandwidth and growth rate of the most unstable wave i.e.

$$(X_1, T_1) = |\Delta|^{\frac{1}{2}}(x, t) \tag{8}$$

We also require the variables

$$(X_2, T_2) = |\Delta|^{\frac{1}{2}}(x, t)$$

and assume that the amplitude $A(X_j, T_j)$ of the wave is a strongly varying function of space and time. The solution my then be developed as an expansion in a small parameter, related to the departure from neutral stability. This method has been formalized by Newell [10] and Weissman [11], and the result is a system of evolution equations which must be satisfied by the wave amplitude at each order of the expansion. The coefficients of the linear terms in these equations are identifiable as derivatives of the characteristic function, H, with respect to its various arguments, the nonlinear terms are specified by the particular problem under consideration.

For the inviscid baroclinic wave models we have

$$HA = 0 \tag{9}$$

$$-H_\sigma A_{T_1} + H_k A_{X_1} = 0 \tag{10}$$

$$\begin{aligned} &i\left(-H_\sigma A_{T_2} + H_k A_{X_2}\right) + \\ &\frac{1}{2}\left(H_{\sigma\sigma} A_{T_1T_1} - 2H_{\sigma k} A_{X_1T_1} + H_{kk} A_{X_1X_1}\right) \\ &= H_\lambda A - N(A) \end{aligned} \tag{11}$$

A number of special cases arise as follows:

i- Marginally unstable wave packets with β or S_2 of O(1)

For an inviscid model, H_σ=0 everywhere on the neutral wave but H_k=0 only at the critical point [11]. The slow scales X_2 and T_2 are not required and the amplitude evolves according to the second order equation

$$\begin{aligned} &\left(\frac{\partial}{\partial T} + C_{g1}\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial T} + C_{g2}\frac{\partial}{\partial X}\right)A = \hat{\sigma}^2 A - NAB \\ &\left(\frac{\partial}{\partial T} + C_{g1}\frac{\partial}{\partial X}\right)|A|^2 = \left(\frac{\partial}{\partial T} + C_{g2}\frac{\partial}{\partial X}\right)B \end{aligned} \tag{12}$$

where $\hat{\sigma}^2$ is the constant of the linear growth rate, N is a positive constant, $B(X_1, T_1)$ is a second order correction to the basic state and

$$C_{g1,2} = \frac{-H_{\sigma k} \pm \sqrt{H_{\sigma\sigma}^2 - H_{\sigma\sigma}H_{kk}}}{H_{\sigma\sigma}}$$

are the group speeds.

Gibbon et al. [6] have shown that the transformation

$$S = \pm 1 - \frac{NB}{\left|\hat{\sigma}^2\right|}, R = \sqrt{2}A \tag{13}$$

and the change of variable

$$\begin{aligned} \xi &= -\sqrt{N}\frac{(X - C_{g1}T)}{C_{g1} - C_{g2}}, \\ \tau &= \left|\hat{\sigma}^2\right| N^{-\frac{1}{2}} \frac{(X - C_{g2}T)}{C_{g1} - C_{g2}} \end{aligned} \tag{14}$$

results in the self-induced transparency equations

$$R_{\xi\tau} = RS \tag{15}$$

$$S_\xi = -\frac{1}{2}|R|^2_\tau$$

with, $S \to \pm 1, R \to 0$ as $|\xi| \to \infty$.

If we assume, in addition that A is real and write

$$R = \phi_\xi, S = \pm\cos\phi \tag{16}$$

then we obtain the sine-Gordon equation

$$\phi_{\xi\tau} = \pm \sin\phi \qquad (17)$$

ii- β (or S_2) = 0

In the absence of sloping endwalls or β, we no longer consider a bandwidth of waves centred at the critical point since this corresponds to k=0. Instead we choose a bandwidth centred about $k_2=O(1)$ which means that $k-k_p=O(\Delta)$ and the appropriate slow time and space scales are now T_1 and X_2. We still have $H_\sigma=0$ and obtain the nonlinear Schrodinger equation with the rules of space and time reversed.

$$iH_k A_{x2} + \frac{1}{2} H_{\sigma\sigma} A_{T_1T_1} = H_\lambda A - N_2 |A|^2 A \qquad (18)$$

iii- Neutral Waves

For neutral waves neither H_σ nor H_k vanishes and replacing $\left(\frac{\partial}{\partial T_1}\right)$ by $\left(\frac{H_\sigma}{H_k}\frac{\partial}{\partial X_1}\right)$ again gives a nonlinear Schrodinger equation.

iv- Marginally unstable wave packets in the inviscid limit

There are basic differences between the linear stability properties for a model in which viscosity is initially set equal to zero and those for a model in which viscosity is retained and then allowed to approach zero. These differences are apparent only when the dispersive effects are present. It is now no longer the case that $H_\sigma=0$ and $H_k=0$ on the stability boundary and the analysis produces an equation of nonlinear Schrodinger type.

4. Comparison with experiments

The quasi-geostrophic models of baroclinic instability behave remarkably like the nonlinear optic problem of the iteration of a rapidly oscillating electric carrier wave with a two-level atomic motion, leading to self induced transparency, where in the viscous limits the NLS, SIT and sine-Gordon equations all appear. It is possible to compare the mechanisms operating in the two systems.

The baroclinic model can also be considered as a system with two energy levels, namely the state of fully developed baroclinic waves and the state of purely zonal flow, as determined by the function S(ξ,τ) which is a measure of the available potential energy of the system. The two extreme states correspond to a maximum (or upper state, when S=+1) or a minimum (or lower state, when S=-1) of available potential energy. The supercritical condition, $\hat{\sigma}^2 > 0$ and the upper state is zonal flow and the lower state being fully-developed waves; the situation is reversed for $\hat{\sigma}^2 < 0$. As in the SIT case the solution is associated with $\hat{\sigma}^2 > 0$ is unstable, the asymptotic state being one of maximum available potential energy and we conclude that only the subcritical case is of importance in an infinite domain.

Such results may well be appropriate for the oceans where typical wavelengths of disturbances are O(50-100) km in a general circulation of 1000 km and it is therefore a good approximation to permit a continuous spectrum of waves to be excited. For laboratory experiments and the atmosphere, typical wavelengths are of an order of magnitude larger and spatially periodic boundary conditions are more appropriate. Exact analytic solutions are known for the sine-Gordon equation in a periodic domain and they take the form of Jacobi elliptic functions. For a finite Josephson transmission line three fundamentally distinct types of solution exist: the plasma, breather and fluxon oscillation. They present respectively an oscillation about zero mean, a bound state oscillation of a vortex- anti-vortex pair repeated reflection at the ends of the transmission line of a fluxon which emerges alternatively as fluxon or antifluxon after reflection. Laboratory experiments have long recognized a tendency for the regular region of baroclinic waves to persist for a considerable ranges of parameter values and this coherent behaviour is due largely to nonlinear effects rather than viscosity. Such behaviour is certainly consistent with soliton solutions described here.

It is customary when discussing the consequences of introducing friction to exactly integrable equations to treat friction as an inhomogeneity and simply add an additional constant term. For the baroclinic wave models described above this is not a mathematically consistent approach, and depending on the amount of friction present and the relative magnitudes of all the non-dimensional parameters present in the problem, dramatically different evaluation equations can arise; an indication of this was apparent in the case (iv) above.

The addition of small fiction effects in the baroclinic wave models give rise to the equations related to the Lorenz. Strange Attractor equations, modified by the presence of a spatial derivative, Alexander [12]; exactly the same behaviour occurs in the nonlinear optics problem [13]. When strong friction is present, an equation arises which resembles the nonlinear Schrodinger equation but which has complex coefficients, usually called the Ginzburg-Landau equation [8]. Both of these equations are known to possess chaotic solutions and their occurrence in this context provides a very interesting field of study in view of the recent interest in the appearance of a periodic solutions when exactly integrable equations are perturbed [14]. We summarize the evolution equations in Table 1.

Table 1. Summary of wave packet behaviour

r \ S_i^β	or $S_i^\beta\} = 0$ or $S_i^\beta\} = O(\Delta)$	or $S_i^\beta\} = O(1)$
r=0	Behaviour unknown: (Equation are NLS with usual space and time dependencies interchanged)	SIT sine-Gordon soliton solutions for infinite domain doubly periodic, Jacobi elliptic solutions for bounded domain. (NLS) Soliton solutions for infinite domain FPU recurrence in bounded domains.
$r = O(\Delta)$	Spatial Lorenz equation under investigation	
$r = O(1)$	Behaviour under investigation equations related to NLS. Some complex coefficients.	Periodic and aperiodic solutions according to parameter values TDGL equation

In summary then the baroclinic wave models provide an excellent example of a new area of physics in which solitons can arise, showing how viscosity alters the form of the exactly integrable equations occurring for an inviscid model. Moreover, the natural development of the (NLS) into the Ginzburg-Landau equation as viscosity is incoherent, implying a change from predictable to unpredictable behaviour, suggests a new way of approaching soliton perturbation problems and a new direction of interest in the study of Hamiltonian systems under perturbation, an area of considerable interest and excitement in dynamical systems theory.

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