IRANIAN JOURNAL OF PHYSICS RESEARCH

Iranian Journal of Physics Research, Vol. 2, No. 3, 2000

Thermal instability in the interstellar medium

J Ghanbari and M Shadmehri
Department of Physics, Ferdowsi University, Mashad, Iran

(Received 10 January 1999 ; in final form 27 March 2000)

Abstract

This study demonstrates how thermal structures in the interstellar medium can emerge as a result of thermal instability. For a two-dimensional case, the steady state thermal structures was investigeted and it was shown that a large class of solutions exist. For a one-dimensional case the conductivity was found to be negligible. The effects of local cooling on the thermal instability were explored in some depth. In this case analytical results for time-dependent cooling function were presented, too. We studied nonlinear wave phenomena in thermal fluid systems, with a particular emphasis on presenting analytical results. When conductivity is proportional to temperature, the behavior of thermal waves is soliton like. For slow thermal waves, approximate analytical results were presented. Extensions of this work are discussed briefly, together with possible astrophysical applications.

Keywords: Interstellar medium, Instabilities, Thermal instability, Thermal Solitons

1. Introduction

Inhomogeneties in interstellar medium (ISM) can be modeled by various instabilities. Thermal instability may be one of the primary causes for a two-phase medium with dense, cool clouds and hot, tenuous intercloud regions. The main motivation behind the studies of radiative condensation has been to explain the formation of dense and cool localized structures in astrophysical and laboratory plasmas, when their masses are less than those required for gravitational contraction. The role of thermal instability has been invoked in quite a number of astrophysical contexts, such as the solar corona [1], broad-line emission regions of active galactic nuclei [2-4], gas in clusters of galaxies and the intergalactic medium [5, 6], and evaporation of accretion disc [7]. We have quoted here only representative works in each field. In addition, thermal instability in its macroscopic form, namely in a fluid of clouds and clumps [8] may also be relevant in the ISM.

The classical thermal instability [9] operates at the thermal pressure of the gas, with heating and cooling appropriate for this thermal gas alone. A more detailed investigation of the growth of condensation in cooling regions has been presented by Schwartz et al [10] who included also the effects of ionization and recombination and by Balbus [11] who examined the effect of magnetic fields. More recent investigations are concentrated on the process of a two-phase medium formation [12] and the dynamics of this medium [13]. Recent progress reports on the theory of thermal instability is given by Balbus [14], and Illarionov & Igumenshchev [15].

By using techniques of pattern theory, Elphick

* ghanbari@science1.um.ac.ir

et al [16] studied thermally bistable media. They proved that for such media, a front, separating the two stable phases, is stationary. Elphick et al [17] have carried these ideas somewhat further by including fluid motions in the model. Ferrara and Shchekinov [18] studied thermal solitons, using phase plane analysis. For a one-dimensional case, using langrangian variable largely facilitates thermal instability analysis [19].

All these authors have presented only numerical or qualitative results. In this paper, our goal is to obtain analytical results as far as possible. Since the full magnetohydrodynamic problem is rather complicated and the chances of finding meaningful analytic solutions seem remote, we used a collection of model equations which approximate the true behavior of the system.

This paper is organized as follows. We begin with a general formulation of the problem in section 2. In section 3 we study steady state solutions. We will see that complex thermal structures can be formed as a result of thermal instability. For one-dimensional case, effects of local or time-dependent cooling functions on the thermal instability are investigated in section 4. In section 5 we study nonlinear thermal waves with particular emphasis on presenting analytical results. For slow thermal waves, we obtain approximate analytical results. A conclusion will be provided in section 6 with a discussion and a summary of our results.

2. General formulation

The macroscopic properties of the interstellar medium are governed by the hydrodynamic equations of motion.

$$\frac{\partial \rho}{\partial t}+\nabla .(\rho v)=0, \tag{1}$$

$$\frac{\partial v}{\partial t}+(v.\nabla)v=-\nabla p/\rho, \tag{2}$$

$$\rho T\left(\frac{\partial s}{\partial t}+(v.\nabla)s\right)=\nabla .(K\nabla T)-L(\rho,T), \tag{3}$$

and equation of state

$$p = nkT \tag{4}$$

where n and s are particle number density and entropy density. $L(\rho,T)$ is cooling-heating function and $K(\rho,T)$ is the heat conductivity of the ISM. Other variables have their conventional meanings. The viscous terms in the hydrodynamic equations have been neglected since they become important only if the velocity v is comparable with the velocity of sound.

The cooling function and heat conductivity are very important; and if we assume that the pressure is constant (see, e.g., [20]) we may write them as a function of temperature (or density). The power law $K \propto T^{\alpha}$ covers most cases of interest. As other physical processes may be involved, we leave α open [21].

It is convenient to work in dimensionless units. It has shown that for the one-dimensional case [19].

$$\frac{\partial T}{\partial t}+\Lambda(T)-\frac{\partial}{\partial m}\left(\frac{K}{T}\frac{\partial T}{\partial m}\right)=0, \tag{5}$$

where $\Lambda(T)$ is the cooling function and m is the Lagrangian mass variable. This is our main equation for thermal instability analysis.

3. Steady state solutions

When the thermal instability is strongly affected by heat conduction, the characteristic time scale of heating and cooling processes is considerably longer than the acoustic time scale. This condition permits the simplification of the fluid equations. It ca be easily shown that from equation (3) we have

$$\frac{\partial u}{\partial t}=u\nabla^2 u+g(u,p), \tag{6}$$

where $K=T^{\alpha}$, $u=T^{\alpha+1}$, and $g(u,p)=-(1+\alpha)T^{\alpha}L(T,p)$. In general case cooling function, $L(T,p)$, is a complicated function [22]. But it is possible to fit the cooling function with these simple forms:

$$g(u,p) = u((u_0-u)^3-\Delta^2(u_0-u)-\beta\log p), \tag{7}$$

or

$$g(u,p=1)=u(-A\sin(Bu+C)), \tag{8}$$

where u_0, Δ, β are parameters, that can be assigned values to fit the cooling function reasonably [16]. We may use equation (8) instead of equation (7) where A, B, and C are parameters that can be obtained by fitting.

In one dimension, for the steady state case from equations (6) and (8) we have

$$\frac{d^2\xi}{dx^2}-\lambda(\xi)=0, \tag{9}$$

where $\xi=Bu+C$ and $\lambda(\xi)=\sin(\xi)$. By introducing $\xi=4\text{Arctan}f(x)$, from equation (9) we have

$$(1+f^2)\frac{d^2f}{dx^2}-2f\left(\frac{df}{dx}\right)^2-f(1-f^2)=0, \tag{10}$$

We see that $\left(\frac{df}{dx}\right)^2=a_4f^4+a_2f^2+a_0$ satisfy in this equation and $a_2=2a_4+1$ and $a_0=a_4$, so we have

$$\int\frac{df}{\sqrt{a_4f^4+(2a_4+1)f^2+a_4}}=x+\text{constant}, \tag{11}$$

where $2a_4+1>0$. Depending on the value of a_4, we have various solutions. We obtain [23]

$$\xi=4\text{Arctan}\sqrt{\frac{1-T_1(x)}{1+T_1(x)}}, \tag{12}$$

where

$$T_1(x)=\begin{cases}\text{cd}(kx,1/k), & a_4>0\\ \tanh x, & a_4=0,\\ k\,\text{cd}(x,k), & 1/4<a_4<0.\end{cases} \tag{13}$$

In these equations cd(x,k) is the Jacobian elliptic function and $k^2=\frac{x_1^2-x_2^2}{x_1^2}$ where

$$x_1^2=\frac{-(2a_4+1)+\sqrt{4a_4+1}}{2a_4},$$
$$x_2^2=\frac{-(2a_4+1)-\sqrt{4a_4+1}}{2a_4} \tag{14}$$

Equations (12) and (13) show that in the static case three different configurations are possible: (1) finite cloud surrounded by hot intercloud gas, (2) hot intercloud layer separating two infinite cold regions, and (3) a periodic sequence of cloud-intercloud regions. Ferrara & Shchekinov [18] described these results qualitatively using phase diagram analysis. But here we present analytical results for steady state thermal structures.

Now after investigating one dimensional steady state thermal structures, we shall study these structures in two or three dimensions. For two or three dimensional cases, obtaining analytical results is much more difficult. In these cases we have

$$\nabla^2\xi=\lambda(\xi) \tag{15}$$

If $\lambda(\xi)$ is linear, we can solve this equation by standard methods. But $\lambda(\xi)$ is a nonlinear function. For some form of this function, analytical solutions exist. For example, if $\lambda(\xi)=-a\xi^2+b\xi$ then [24],

$$\xi(x,y)\cong\frac{3b}{2a}\frac{1}{\cosh^2(\frac{\sqrt{b}}{2}y)}\left[1+\frac{\varepsilon\cos(\frac{\sqrt{5b}}{2}x)}{\cosh(\frac{\sqrt{b}}{2}y)}\right], \tag{16}$$

where a and b are constant parameters and ε is a small parameter. This solution shows a chain of two dimensional standing thermal solitons.

For two dimensional cases, we can investigate solutions of equation (15) qualitatively. From this equation we have

$$\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\xi}{\partial y}\right)^2=F(\xi), \tag{17}$$

where $F(\xi)=2\int\lambda(\xi)d\xi$. From this equation it is clear that the variety of steady state solutions in higher dimensions is, richer than the one dimensional case. Equation (17) defines a surface in $\frac{\partial\xi}{\partial x},\frac{\partial\xi}{\partial y},\xi$ -space. Such a surface which we designate by S here is the locus of points whose coordinates satisfy this equation. In fact in one dimension, equation (17) defines curves in phase space [18]. These curves are projections of surface S on one of the coordinate planes (that is, $\frac{\partial\xi}{\partial x}-\xi$ or $\frac{\partial\xi}{\partial y}-\xi$ planes). So any curve on S represents one of the solutions. We may consider various curves on S which correspond to standing nonlinear temperature waves. Their spatial period is determined by the specific form of the curves on S and the form of cooling function.

4. Thermal instability when K → 0

Nonlinear equations similar to equation (5) arose

in a large number of studies devoted to various problems of nonlinear waves, instabilities, and structures in dissipative media (see, e.g., [25]). In section 3 we have investigated steady state (time-independent) thermal structures. But solving equation (5) in general case is difficult. In this section we consider a simple case. If the heat conduction term in this equation can be neglected, the nonlinear dynamics of the thermal instability is determined solely by the form of the cooling function and by initial conditions. We have

$$\frac{\partial T}{\partial t}+\Lambda(T)=0 \tag{18}$$

Now we investigate effects of local cooling on the thermal instability when heat conduction can be neglected. We assume a local cooling term as -f(m), so

$$\frac{\partial T}{\partial t}=-\Lambda(T)+f(m) \tag{19}$$

If $\Lambda(T)=1-T$ and $T(m,t=0)=T_0(m)$, then from equation (19) we have

$$T(m,t)=1-f(m)+(f(m)+T_0(m)-1)e^t. \tag{20}$$

It is clear from equation (20) that, the regions with $T_0(m)>1-f(m)$, will be heated, while those with $T_0(m)<1-f(m)$ will be cooled. The regions with $T_0(m)<1-f(m)$ cool until the temperature at some point (say, m_*) becomes zero. At this point, the gas density becomes infinitely large (explosive condensation). This effect takes place at the time moment

$$t_*=\ln\left[\frac{1-f(m_*)}{1-f(m_*)-T_0(m_*)}\right] \tag{21}$$

Now we consider another form for cooling function: $\Lambda(T)=T-T^2$. Here we have two equilibrium points, $T=0$ and $T=1$. If we introduce $4f(m)-1=c(m)$, we may have $c(m)<0$ or $c(m)>0$ or $c(m)=0$. For each of these cases, equation (19) can be solved. For example, if $c(m)>0$ the gas temperature behaves as

$$T(m,t)=\frac{1}{2}\left\{1+\sqrt{c(m)}\,\frac{\tan\left(\frac{\sqrt{c(m)}}{2}t\right)+\frac{2T_0(m)-1}{\sqrt{c(m)}}}{1-\frac{2T_0(m)-1}{\sqrt{c(m)}}\tan\left(\frac{\sqrt{c(m)}}{2}t\right)}\right\} \tag{22}$$

and it becomes infinite at m_* at the time moment

$$t_*=\frac{2}{\sqrt{c(m_*)}}\arctan\left(\frac{c(m_*)}{2T_0(m_*)-1}\right) \tag{23}$$

Simultaneously, the gas density becomes zero at this point. It seems when we consider the local cooling function, we deal with situations where the solution blows up in a finite time leading to a catastrophic instability.

Now we consider the time-dependent cooling function as g(t). So

$$\frac{\partial T}{\partial t}=-\Lambda(T)-g(t) \tag{24}$$

For $\Lambda(T)=T^2-T$ and $T=\frac{1}{u}\frac{\partial u}{\partial t}$ this equation becomes

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial u}{\partial t}+g(t)u=0 \tag{25}$$

where by introducing $u=e^{\frac{1}{2}}\xi$, we have

$$\frac{\partial^2 \xi}{\partial t^2}+\left(g(t)-\frac{1}{4}\right)\xi=0 \tag{26}$$

Depending on the specific form of g(t), this equation may have periodic solution that becomes zero at some points. For example, if $g(t)=\cos(t)$, equation (26) becomes Mathieu's equation. So for time-dependent cooling function, we may have explosive rarefaction or condensation. In fact this form of cooling function or local cooling function do not remove these explosive behaviors.

5. Nonlinear thermal waves

It has been shown that equation (6) has traveling wave solutions [16]. Ferrara & Shchekinov [18] investigated these nonlinear thermal waves, by using phase diagram analysis. In this section, we shall show that for $K\propto T$, equation (5) becomes

well-known equations. These nonlinear thermal waves are very important. The key role played by the fronts between the cold and the hot gas in thermal systems has been recognized by Zel'dovich & Pikel'ner [26], who formulated a phase rule giving the value of the ambient pressure necessary for stationary of a front between the two stable phases.

But stability of thermal waves remained as an open question. For $K \propto T$, we shall find solitary thermal waves. In this case from equation (5), we have

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial m^2} - \Lambda(T) \tag{27}$$

There are a lot of works on this nonlinear diffusion equation. The general properties of equation (27) have been systematically investigated by Aronson & Weinberger [27]. They proved that from a wide class of initial data, the solution of this equation will converge to a local travelling wave with a definite speed. It is also pointed out that there exists a critical speed C^* for the waves, which can be estimated by

$$2\sqrt{\left(\frac{d\Lambda}{dT}\right)_{T=0}} \le C^* \le 2\sqrt{L}, L \equiv \sup\frac{\Lambda(T)}{T} \tag{28}$$

We can choose appropriate cooling function [22]. For $\Lambda(T)=T(T-1)$, this is the well-known Fisher equation; and for $\Lambda(T)=T(k-T^{\alpha})(T^{\alpha}-1)$, where α is a parameter and $0 \le k \le \frac{1}{2}$, it becomes Huxley equation. These equations have solitary wave solutions.

Now we obtain exact traveling wave solutions of equation (27) for $\Lambda(T)=\beta(T^p-T^q)$, where β, p and q are constant parameters. If we let

$$T(m,t)=T(y),\ y=m-ct, \tag{29}$$

where c is an undetermined constant, then equation (27) becomes

$$c\frac{dT}{dy} + \frac{d^2T}{dy^2} = \Lambda(T) \tag{30}$$

We assume $\frac{dT}{dy} = aT + bT^n$, where a, b and n are undetermined parameters. In fact, this ansatz is the famous Bernoulli equation. We can rewrite equation (27) with the help of this ansatz. It gives the value of undetermined parameters. Equation (27) has traveling wave solutions for the following cases.

(a) For p=1 and $\beta(q+1)>0$, we obtain

$$T(m,t)=\left\{\frac{1}{2}\tanh\left[\frac{q-1}{2}\sqrt{\frac{\beta}{2(q+1)}}\left(m+\sqrt{\frac{\beta}{2(q+1)}}(q+3)t+c_0\right)\right]+\frac{1}{2}\right\}^{\frac{2}{q-1}} \tag{31}$$

where c_0 is an arbitary constant. This ia a solitary wave solution.

(b) For 2p=q+1 and $\beta(q+1)>0$, we have

$$T(m,t)=\left\{\frac{1}{2}\tanh\left[\frac{(q-1)\beta}{2}\sqrt{\frac{1}{2(q+1)}}\left(m+\sqrt{\frac{2\beta}{q+1}}t+c_0\right)\right]+\frac{1}{2}\right\}^{\frac{2}{q-1}} \tag{32}$$

For $\Lambda(T)=g\sin T$, we obtain the slow thermal wave solution for equation (27). We express the solution in the form of expansion in terms of a small parameter $c/L=\varepsilon$ as

$$T(m,\varepsilon) = T_0(m) + \varepsilon T_1(m) + \varepsilon^2 T_2(m) + \ldots \tag{33}$$

and substitute $T(m,\varepsilon)$ into equation (27). So we have

$$\frac{d^2T_0}{dm^2} - g\sin T_0 = 0 \tag{34}$$

and we have solved this equation

$$T_0(m) = 4\mathrm{Arctan}\left(\frac{1-h}{1+h}\right)^{\frac{1}{2}} \tag{35}$$

where $h = h(\sqrt{g}\,m)$ and

$$h(y) = \begin{cases} \mathrm{sn}(ky, 1/k), & k>1 \\ \tanh y, & k=1 \\ k\,\mathrm{sn}(y,k), & 1>k>0 \end{cases} \tag{36}$$

The equation satisfied by $T_1(m)$ is

$$\frac{d^2T_1(m)}{dm^2} - gT_1\cos T_0 = -\frac{dT_0}{dm} \tag{37}$$

When k=1, the solution of this equation becomes

$$T_1(m) = \frac{1}{\sqrt{3g}} \tanh^2\left(\sqrt{\frac{g}{2}}m\right)\left(\cosh^2\left(\sqrt{\frac{g}{2}}m\right)+2\right) \quad (38)$$

Calculation of the functions $T_2(m)$, $T_3(m)$, ... can similarly be done. For fast thermal waves, equation (27) can be solved [28].

6. Summary

Thermal instability seems to play an important role in the process of formation of different thermal phases in ISM. In this paper we studied thermal instability in various cases. Steady state thermal structures have been investigated in detail. For one-dimensional case, we presented analytical results and it has been shown that in this case three different time-independent thermal structures are possible. We have hot (or cold) cloud surrounded by cold (or hot) intercloud gas; or a periodic sequence of hot-cold regions.

For two or three dimensional cases, presenting analytical results for the steady state thermal structures is difficult. In fact like the one-dimensional case, the behaviour of solutions depends on the form of cooling function and for some forms of this function, analytical solutions exist. But in this case we can study steady state solutions qualitatively. It has been shown that the variety of steady solutions is richer than that for the dimensional case. There remains an interesting question about the conditions for realization of steady state structures, in problems with different initial conditions.

When heat conduction is negligible, we can investigate the nonlinear dynamics of the thermal instability analytically. In fact, in this case behaviour of thermal instability is determined only by the form of the cooling function and by initial conditions. We studied the effects of local and time-dependent cooling function on the thermal instability. In these cases, we obtained explosive condensation or rarefaction as a result of thermal instability. Analytical results for these have been presented.

It has been shown that solitary wave solutions are admitted by the equation describing thermal system. In fact when conductivity is proportional to temperature, this equation becomes a well-known equation depending on the form of cooling function. These equations have solutions and they are stable. Therefore, stable thermal waves (thermal solitons) can be formed in the ISM as a result of thermal instability. For other cases (i.e. $K \propto T^{\alpha}$ where $\alpha \neq 1$), we have thermal waves which may be unstable. This is an open question. For slow thermal wave, we obtained analytical results.

These thermal solutions may lead to a different picture of the ISM. Since they are stable, time-dependent patterns which propagate can be formed. Highly ionized species are detected either in the disk or in the halo. Thermal solitons may produce these patterns. A serious extension of the present work would be the examination of thermal waves in two or three dimensions. We have concentrated on the study of stationary waves in one spatial dimension. Within this class of waves, a great variety of solutions exists. However, more general types of wave solutions should be studied.

References

1. J T Karpen, S K Antiochos, J M Picone, & R B Dahlburg, 493. *ApJ*, **338**, 1989.
2. R McCray, in Active Galactic Nuclei, ed. C Hazard & S Mitton (*Cambridge: Cambridge Univ. Press*), **227**, 1979.
3. J H Krolik, 148, *ApJ*, **325**, 1988.
4. C F McKee, & M C Begelman, *ApJ*, **358**, 392, 1990.
5. A G Doroshkevich, & Ya B Zel'dovich, *Soviet Phys.-JETP*, **53**, 405, 1981.
6. M Hattori, & A Habe, *MNRAS*, **242**, 399, 1990.
7. M Czerny, & A R King, *MNRAS*, **236**, 843, 1989.
8. B G Elmegreen, *ApJ*, **344**, 306, 1989.
9. G B Field, *ApK*, **142**, 531, 1965.
10. J Schwartz, R McCray, & R F Stein, *ApJ*, 673, **175**, 1972.
11. S A Balbus, ApJ, L79, **303**, 1986.
12. I Aranson, B Meerson, & P V Sasorov, *Phys. Rev. E*, **47**, 4337, 1993.
13. V Aharonson, O Regev, & N Shaviv, *ApJ*, **426**, 621, 1994.

14. S A Balbus, The Physics of the Interstellar Medium and Intergalactic Medium, ASP Conf. Ser. 80, ed. A Ferrara, C F McKee, C Heiles & P R Shapiro, 1995, p. 328.
15. A F Illarionov, & I V Igumenshchev, *MNRAS*, **298**, 909, 1998.
16. C Elphick, O Regev, & E A Spiegel, *MNRAS*, **250**, 617, 1991.
17. C Elphick, O Regev, & N Shaviv, *ApJ*, **392**, 106, 1992.
18. A Ferrara, & Yu Shchekinov, *ApJ*, **417**, 595, 1993.
19. B Meerson, *ApJ*, **347**, 1012, 1989.
20. R Graham, & W D Langer, *ApJ*, **179**, 469, 1973.
21. L L Cowie, & C F McKee, *ApJ*, **211**, 135, 1977.
22. S Lepp, R McCray, J M Shull, D T Woods, & T Kallman, *ApJ*, **288**, 58, 1985.
23. I S Gradshteyn, & I M Ryzhik, Table of Integrals, *Series and Products* (Academic Press), 1980.
24. A B Mikhailovskii, in Nonlinear Phenomena in Plasma Physics and Hydrodynamics (Mir Publishers), 1986.
25. B Straughan, Explosive Instabilities in Mechanics (Springer), 1998.
26. Ya B Zel'dovich, & S B Pikel'ner, *Soviet Phys. JETP*, **29**, 170, 1969.
27. D G Aronson, & H F Weinberger, in Lecture notes in mathematics, Vol **446** ed. J. A. Goldstein (Springer), 1975.
28. Xu Gang, S Changqing, & L Lei, *J. Math. Phys.*, **26**, 1566, 1985.