Mean field theory of high T_c cuprate superconductivity

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Abstract

Two decades ago the epoch making discovery of high T_c cuprate superconductivity by Bednorz and Müller shocked the world's superconductivity community. However, already in 1979 and 1980, the first heavy fermion superconductor $CeCu_2Si_2$ and organic superconductor $(TMTSF)_2PF_6$ have been discovered respectively. Also we know now that all these superconductors are unconventional and nodal. Further the quasiparticles in the normal state in these systems are Fermi liquids and the superconducting states are described in terms of generalized BCS wave function. Also the pseudogap phase in underdoped high T_c cuprates is described in terms of d-wave density wave. This implies necessarily that the superconductivity in underdoped cuprates is gossamer (i.e. d-wave superconductivity coexists with d-wave density wave). We shall present some quantitative tests of these new concepts, notions and ideas.

1. Introduction

We have to go back to 1979 when the first unconventional superconductor was discovered in the heavy fermion system $CeCu_2Si_2$ by Steglich et al.[1]. These systems are called heavy fermion systems, since the effective mass m^* of the quasiparticles is 10^2-$10^3 \times m$, where m is the free electron mass. This extremely large mass m^* is due to the strong Coulomb correlation and the quasiparticles are well described in terms of Landau's theory of Fermi liquid [2-4]. In particular, the superconductivity in such a system is very surprising due to the well-known antagonism between magnetism and superconductivity [5]. As we shall see shortly, the superconductivity in $CeCu_2Si_2$ is unconventional and nodal [6], though we do not know the symmetry of its gap function yet. In a rapid succession, many more heavy fermion superconductors have been discovered, UPt_3, UBe_{13}, URu_2Si_2, UPd_2Al_3, UNi_2Al_3, $CeCoIn_5$…. In 1980, Jerome et al. [7] discovered the first organic superconductor in Bechgaard salt $(TMTSF)_2PF_6$. Again we cannot count more then 20 organic superconductors, most of them unconventional[8]. Surprisingly, until this century, these superconductors have been considered as conventional s-wave ones.

Than in 1986 the first high T_c superconductor $La_{2-x}Ba_xCuO_4$ was discovered by Bednorz and Müller [9]. Of course this discovery was welcomed with enthusiasm and exaltation. On the other hand, it put the theoretical community into confusion as if Pandora's box were open. This situation is beautifully recorded in a textbook by Charles Enz [10]. Perhaps the most influential were Anderson's dogmas [11]. Anderson proposed that high T_c superconductivity arises in the middle of doped Mott insulator. As the simplest model, he took 2 dimensional one band Hubbard model on a square lattice. Also as a model for the superconducting wave function, he proposed the resonating valence bond (RVB) state. In spite of tremendous and relentless studies on Anderson's dogmas, we still do not have a simple picture of high T_c cuprates. Rather we shall follow Laughlin's lead [12]. We shall show that high T_c cuprate superconductivity in the underdoped region is considered as "Gossamer superconductivity" (i.e. d-wave superconductivity in coexistence with d-wave density wave) [13,14].

In the meanwhile from the perturbative study of the Hubbard model, d-wave superconductivity in high T_c cuprates is predicted [15]. Also the renormalization group study of 2 dimensional fermion systems [16-18] indicates, that a.) they are Fermi liquids and b) the possible ground state is related to the infrared instability of the two particle (and two hole) channel or the particle-hole channel. They correspond to conventional or unconventional BCS superconductors and conventional or unconventional density waves. As seen from the summary talk by Randeira [19] at M2S-Rio Conference on May 2003, the questions a) if the quasiparticles are

Fermi liquids and b) if the superconductivity is of BCS type echo through the corridor of high T_c cuprate superconductors. But in 1994 these questions have been answered already at least theoretically.

The single crystals of optimally doped LSCO, YBCO, and thin films Bi2212 become available around 1992. Finally d-wave symmetry of high T_c cuprate superconductivity was established through the powerful angle resolved photoemission spectroscopy (ARPES) [20], and elegant Josephson interferometry [21,22] among many other experiments.

Therefore a few people started to study the BCS theory of d-wave superconductors. For example, the impurity scattering in d-wave superconductors produced many surprising effects. In 1993, Patrick Lee[23] discovered the universal heat conduction. In the cleanest single crystals of d-wave superconductors like high T_c cuprates, the electronic thermal conductivity at low temperatures is linear in temperature T. Further in the limit of small impurity scattering, κ/T is independent of Γ (the quasiparticle scattering rate) and given by

$$\lim_{T,\Gamma \to 0} \frac{\kappa}{T} = \frac{\kappa_{00}}{T} = \frac{k_B^2}{3\hbar} \frac{\upsilon}{\upsilon_2} n, \tag{1}$$

where $\upsilon/\upsilon_2 = E_F/\Delta(0)$ and n is the hole density, E_F is the Fermi energy (~ 5000 K) and $\Delta(0)$ is the maximum value of the energy gap at T=0 K. Also for d-wave superconductivity, we take

$$\Delta(k) = \Delta(T)\cos(2\phi) = \Delta(T)(\hat{k}_x^2 - \hat{k}_y^2). \tag{2}$$

Also we have shown that the thermal conductivity increases with Γ [24]. This counter intuitive behaviour is established by Taillefer et al[25]. More recently the universal heat conduction is generalized to other nodal superconductors[26]. In 1997, May Chiao et al.[27,28] measured the thermal conductivities of single crystals of optimally doped Bi2212 and YBCO below 1 K. From these experiments they have extracted $\Delta(0)/E_F = 1/10$ and 1/14, respectively. These numbers indicate the followings[13,19]:

a) High T_c cuprate superconductivity is in the BCS limit and very far away from the Bose-Einstein condensation limit.

b) Making use of the Ginzburg criterion, the effect of superconducting fluctuations is at most of the order of $\Delta(0)/E_F \sim (\xi p_F)^{-1}$. Therefore this appears to rule out models with large superconducting fluctuations in the underdoped region of high T_c cuprates [30,31]. Very recently, the Josephson current between underdoped and optimally doped superconductor (YBCO) was measured [32]. The result is fully consistent with the one expected for BCS superconductors[33]. There will be no preformed pair or large phase fluctuation.

c) There are hundreds of quasiparticle bound states around a single vortex in d-wave superconductors [34,35]. These bound states are analogues of the well-known Caroli de Gennes Matricon (CdGM) bound states around a single vortex in s-wave superconductors [36,37].

A few years ago it was claimed that there was no bound state around a vortex in d-wave superconductors [38-41]. Unfortunately in all these calculations, the authors have assumed $\Delta \sim E_F$ in order to make calculations easier, resulting in completely unphysical conclusion. This is deplorable that still many authors in the high T_c community practice this unphysical assumption $\Delta \sim E_F$. In order to eliminate these misleading and useless papers from the superconducting community, extreme vigilance has to be exercised. For example if you exclude all irrelevant references from the otherwise excellent review by Hussey [42], we can conclude that all experiments on single crystals of high T_c cuprates are fully consistent with the BCS theory of d-wave superconductivity.

2. The Volovik effect

In 1993, Volovik[43] has shown how to calculate the quasiparticle density of states in the vortex state in d-wave superconductors. A surprising $\sqrt{H}$ dependence of the specific heat on the magnetic field (H) was confirmed experimentally in single crystals of YBCO[44,45], LSCO [46] and Sr$_2$RuO$_4$[47,48]. This semiclassical approach has been extended into a variety of directions [49-53]: a) for the scaling relations and the thermal conductivity, b) for arbitrary field orientation and c) for a variety of nodal superconductors. Perhaps we should point out that the expression of the thermal conductivity in Ref. [50] is incorrect, which is corrected in Ref. [53]. Also the authors of Refs. [51,52] have introduced a circular Fermi surface rather than a cylindrical one [54,55], and therefore their predictions are unreliable.

Also in Refs. [53,54] the concept of the clean limit and the superclean limit for unconventional superconductors are introduced for the first time. Further these analysis suggests that the magnetothermal conductivity in the vortex state should provide the most sensitive test for the nodal structure of the superconducting gap function $\Delta(k)$.

Since 2001 in a brilliant series of experiments, Izawa et al. have successed in identifying the gap functions of Sr$_2$RuO$_4$ [56], CeCoIn$_5$ [57], κ-(ET)$_2$Cu(NCS)$_2$ [58], YNi$_2$B$_2$C [59], PrOs$_4$Sb$_{12}$ [60,61] and UPd$_2$Al$_3$ [26,62]. These gap functions are shown in figure 1. Note that $\Delta(k)$ in CeCoIn$_5$ and κ-(ET)$_2$Cu(NCS)$_2$ belongs to d-wave superconductors as in high T_c cuprates, $\Delta(k)$ in Sr$_2$RuO$_4$ and UPd$_2$Al$_3$ are chiral f-wave and g-wave though there are some controversy.

The $s+g$-wave for YNi$_2$B$_2$C and $p+h$-wave for PrOs$_4$Sb$_{12}$ are unique in the sense that these are the only order parameters consistent with magnetothermal conductivity data given in Refs. [59-61] This remarkable success is possible due to the facts, that a) adequate theoretical developments following Volovik's work, b)
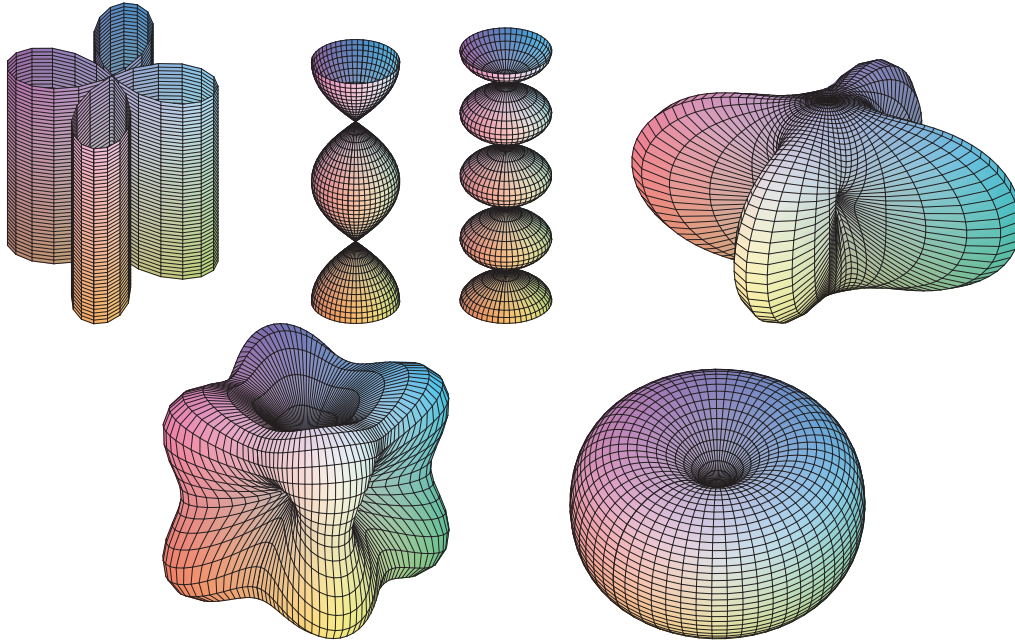
Figure 1. (color online) Order parameters from top left: *d*-wave - high- T_c cuprates, CeCoIn$_5$, κ-(ET)$_2$Cu(NCS)$_2$; chiral *f*-wave – Sr$_2$RuO$_4$; *g*-wave – Upd$_2$Al$_3$; *s+g*-wave – Yni$_2$B$_2$C; *p+h*-wave, PrOs$_4$Sb$_{12}$ A-phase; *p+h*-wave, PrOs$_4$Sb$_{12}$ B-phase.

the availability of high quality single crystals and c.) the medium low temperature facility which operates at 10 mK<*T*<4 K. Also all these new superconductors appear to be described in terms of the weak-coupling BCS theory.

3. D-wave density waves

In parallel to unconventional superconductors, it was natural to consider unconventional density waves [63]. Recently several people have proposed that the pseudogap phase in the *x-T* phase diagram of high T_c cuprate superconductors is *d*-wave density wave (*d*DW) [64-66]. More recently the giant Nernst effect[67-79] and the angle dependent magnetoresistance [70] in the pseudogap region were interpreted in terms of *d*-wave density waves [13,71,72]. Note that *d*DW we are considering is the mean-field solution of 3D systems unlike earlier ones considered in Refs. [64-66]. These authors considered a 2D system on a square lattice and *d*DW has minuscule loop currents with Z_2 symmetry, which is the descendant of the flux phase [73,74]. But it is easy to show that such a purely 2D construction is unstable in a real 3D environment like in high T_c cuprates.

Instead our mean-field solution [63] possesses the $U(1)$ gauge symmetry and in principle our UDW can slide in the presence of an electric field as shown in the low temperature phase of α-(BEDT-TTF)$_2$KHg(SCN)$_4$ [75-77]. Unconventional density waves in quasi-one dimensional systems have been reviewed in Ref. [67]. On the other hand, in quasi 2D systems like high T_c cuprates, CeCoIn$_5$ and κ-(ET)$_2$ salts, *d*-wave density waves appears to be the most relevant. For example,

CeCoIn$_5$ belongs to heavy fermion systems [79]. It has the layered structure as high T_c cuprates and d-wave superconductivity with T_c =2.3 K [57]. More recently the giant Nernst effect and the angle dependent magnetoresistance in the pseudogap phase of CeCoIn$_5$ have been reported [80,81], which were successfully interpreted in terms of *d*DW[72,82,83]. In all of these analysis, the Landau quantization of the quasiparticle energy spectrum in a magnetic field as first discussed by Nersesyan et al. [84,85] played the crucial role. Let us consider a *d*DW in a magnetic field B tilted from the *c* axis by an angle θ. In the absence of magnetic field the quasiparticle spectrum in *d*DW is given by

$$(E(k) + \mu)^2 = \upsilon^2(k - k_F)^2 +$$
$$\Delta^2 \cos^2(\phi) \simeq \upsilon^2(k - k_F)^2 + \upsilon_2^2 k_\perp^2, \quad (3)$$

where $\upsilon_2 / \upsilon = \Delta / E_F$ with υ, Δ and μ the Fermi velocity in the *a-b* plane, the *d*DW gap and the chemical potential, respectively. Also k and $k_\perp$ are the radial and prehelical component of the quasiparticle wavevector. Further the second equation in eq. (3) is valid in the vicinity of the Dirac cones at $(k,\phi) = (k_F, \pm\pi/4)$ etc. In the presence of magnetic field, the energy spectrum becomes

$$(E_n + \mu)^2 = 2n\upsilon\upsilon_2 e \mid B \cos(\phi) \mid, \quad (4)$$

with n=0, 1, 2, 3…. Also except for the n=0 state, which is nondegenerate, all other states are doubly degenerate. Here we have neglected the quasiparticle motion parallel to the *c* axis for simplicity. As already discussed in Ref. [84,85], the magnetothermodynamics of *d*-wave density waves is already obtained. For example the free energy is given by
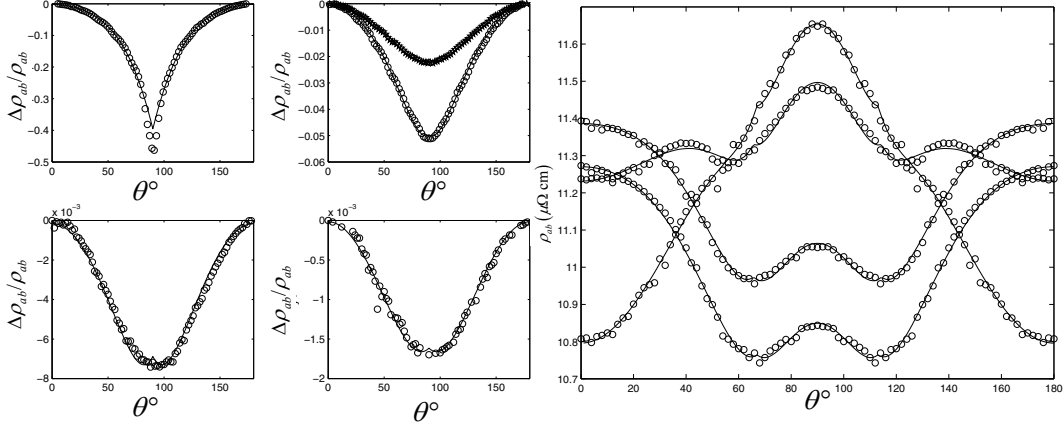
Figure 2. The relative change of the in-plane magnetoresistance of $Y_{0.68}Pr_{0.32}Ba_2Cu_3O_7$[70] is plotted in the left panel as a function of angle θ at H=14 T for T=52 K (top left), 60 K and 65 K (top right), 75 K (bottom left) and 105 K (bottom right). The solid line is fit based on eq. 9. In the right panel, the magnetoresistance of $CeCoIn_5$[81] is shown as a function of angle θ at T=6 K for 4 T, 5 T, 8 T and 10 T from bottom to top at θ=0. The solid line represents our fit based on eq. (9).

$$F = -\frac{8eBT}{\pi}\sum_{n,\pm}\ln\left(1+e^{\beta E_n^{\pm}}\right) = -$$
$$\frac{16eBT}{\pi}\Big(\ln(1+e^{-\zeta_0})+ \qquad\qquad (5)$$
$$\left[2\zeta_0+\ln(2(\cosh(\zeta_0)+\cosh(x_1)))\right]+...\Big),$$

where $\zeta_0 = \beta\mu$, $x_1 = \beta\sqrt{2\upsilon\upsilon_2 e\,|B\cos(\theta)|}$, $\beta = 1/k_B T$. From this the magnetization is given by

$$M = -\frac{F}{B} =$$
$$\frac{16eT}{\pi}\Big[\ln(1+e^{-\zeta_0})+2\zeta_0+\ln(2(\cosh(x_1) \qquad (6)$$
$$+\cosh(\zeta_0)))+\frac{1}{2}x_1\frac{\sinh(x_1)}{\cosh(x_1+\cosh(\zeta_0))}\Big],$$

or the diamagnetic susceptibility

$$\chi = \frac{32\upsilon\upsilon_2 e^2}{\pi T}\left(\frac{1}{x_1}\frac{\sinh(x_1)}{\cosh(x_1)+\cosh(\zeta_0)}+\right.$$
$$\left.\frac{1+\cosh(x_1)\cosh(\zeta_0)}{\cosh(x_1)+\cosh(\zeta_0)}\right). \qquad (7)$$

For $x_1 \gg 0$, this reduces to

$$\chi = \frac{32\upsilon\upsilon_2 e^2}{\pi T}\sum_n x_n^{-1} = \frac{16\sqrt{2}e^{3/2}\sqrt{\upsilon\upsilon_2}}{\sqrt{|B\cos(\theta)|}}\zeta\left(\frac{1}{2}\right)\simeq \quad (8)$$
$$-33.036 e^{3/2}\sqrt{\upsilon\upsilon_2}\,|B\cos(\theta)|^{-1/2}.$$

This diamagnetic susceptibility diverges like $B^{-1/2}$ in the $T\to 0$ limit. As already discussed elsewhere, the angle dependent magnetoresistance and the giant Nernst effect are given by[71,72]

$$\sigma_{xx} = \sum_n \sigma_n sech^2\left(\frac{\beta E_n}{2}\right), \qquad (9)$$

where σ_n's are the level conductivities weakly depending on temperature and magnetic field, and

$$\alpha_{xy} = -\frac{S}{B\sigma_{xx}}, \qquad\qquad (10)$$

where

$$S = 2eB\Big[\ln\left(1+e^{-\zeta_0}\right)+2\sum_n\big(-\zeta_0+\ln(2(\cosh(x_n)+$$
$$\cosh(\zeta_0)))+\frac{x_n(e^{\zeta_0}+\sinh(x_n))+\zeta_0\sinh(x_n)}{\cosh(x_n)+\cosh(\zeta_0)}\Big)\Big] \qquad (11)$$

In figure. 2 we compare the expression (9) with the angle dependent magnetoresistance observed in underdoped high T_c cuprate $Y_{0.68}Pr_{0.32}CuO_4$ [70] and $CeCoIn_5$ [81]. In the analysis of the data of $CeCoIn_5$, it is necessary to include υ_c, the Fermi velocity along the c-axis with $\upsilon_c/\upsilon =1/2$. Clearly these fittings are excellent, and from these we can deduce $\Delta = 360$ K, $\upsilon = 2.3\times 10^7$ cm/s and $\mu = 40\sim 60$ K for $Y_{0.68}Pr_{0.32}CuO_4$ and $\Delta = 45$ K, $\upsilon = 3.3\times 10^6$ cm/s and $\mu = 8.4$ K for $CeCoIn_5$. These values are very reasonable for these systems. We have already reported similar fittings of the giant Nernst effect [71,83]. In summary, the angle dependent magnetoresistance and the giant Nernst effect appear to provide clear signature of unconventional density waves [78].

4. Gossamer superconductivity
How to understand the phase diagram of high T_c cuprate superconductivity has been the central issue from the beginning. We have seen that Anderson's dogmas and RVB state [11] cannot solve this question. Bob Laughlin [12] proposed an alternative wave function and called the "Gossamer superconductivity". Actually the simplest interpretation of Gossamer superconductivity is d-wave superconductivity coexisting with other order parameter [13,14]. Then what is the pseudogap phase? As we have seen in the proceeding section, the pseudogap phase should be d-wave density wave. Therefore within the
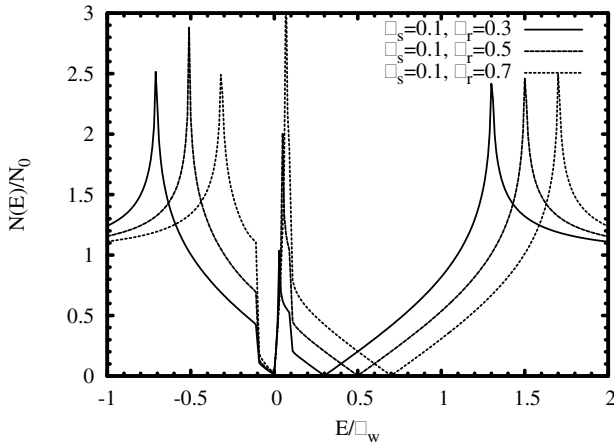
Figure 3. The density of states of a Gossamer superconductor is plotted for various Δ's.

context of high T_c cuprate superconductivity and related systems like $CeCoIn_5$ and κ-$(ET)_2$ salts, we can define that Gossamer superconductivity is the d-wave superconductivity in coexistence with d-wave density wave.

In particular, Gossamer superconductivity possesses two order parameters Δ_1 and Δ_2 for d-wave density wave and d-wave superconductor. Then the Nambu-Gor'kov Green's function is given by

$$G^{-1}(\omega,k) = \omega - \mu\sigma_3 - \tilde{\xi}(k)\rho_3\sigma_3 +$$
$$\Delta_1 e^{-i\phi_1\rho_3}\sigma_3\rho_1 - \Delta_2 e^{-i\phi_2\sigma_3}\sigma_1\rho_3, \qquad (12)$$

where $\tilde{\xi}(\mathbf{k}) = (\xi(\mathbf{k}) - \xi(\mathbf{k+Q}))/2$, though in the followings we drop the tilde off from $\xi(\mathbf{k})$. Also ρ_i and σ_i are Pauli matrices which operate on space formed by $(c_{\mathbf{k},\alpha}^+, c_{-\mathbf{k},-\alpha}, c_{\mathbf{k+Q},\alpha}^+, c_{-\mathbf{k-Q},-\alpha})^+$. Here ϕ_1 and ϕ_2 are the phases of Δ_1 and Δ_2, respectively. These U(1) gauge variables describe the sliding motion of UDW and the supercurrent in unconventional superconductors as well. Then the quasiparticle density of states is given by

$$\frac{N(E)}{N_0} =$$
$$Re\left\langle\left|\frac{E}{\sqrt{(\sqrt{E^2-\Delta_2^2 f^2}-\mu)^2-\Delta_1^2 f^2}}\left(1\mp\frac{\mu}{\sqrt{E^2-\Delta_2^2 f^2}}\right)\right|\right\rangle,$$
$$(13)$$

where $+$ and $-$ stands for $E > 0$ and $E < 0$, respectively, $f = \cos(2\phi)$ and $\langle...\rangle$ means average over ϕ. These are plotted in figure 3. Here we limit ourselves for $\Delta_2 \ll \Delta_1$. Then we see the superconducting energy gap produces an extra structure in the energy region $|E| \sim \Delta_2$.

The low energy effective Hamiltonian for the gossamer superconductivity is readily constructed from the general Hubbard model which includes the nearest neighbor Coulomb interaction and the exchange term

[63]. From this we derive a coupled equation

$$\lambda_1^{-1} = 4\pi T \sum_{n\geq 0} Re\left\langle f^2 d^{-1/2}\right\rangle, \qquad (14)$$

$$\lambda_2^{-1} = 4\pi T \sum_{n\geq 0} Re\left\langle f^2\left(1-\frac{\omega_n}{\sqrt{\omega_n^2+\Delta_2^2 f^2}}\right)d^{-1/2}\right\rangle, \qquad (15)$$

where

$$d = (\sqrt{\omega_n^2+\Delta_2^2 f^2}-i\mu)^2 + \Delta_1^2 f^2. \qquad (16)$$

Here $\omega_n = 2\pi T(n+1/2)$ is the Matsubara frequency and the n sums have to be cut off at $n \simeq E_0/(2\pi T)$ and $E_0 \sim 5000K$. Also λ_1 and λ_2 are the dimensionless coupling constants. Earlier we have shown that if both the superconductivity and the density wave are s-wave, there will be no coexistence. Even if just one of them is s-wave, the coexistence appears to be very unlikely. In other words in order to have the gossamer superconductivity, both the superconducting and the density wave are to be unconventional[86]. Now assuming λ_1 and λ_2 are independent of μ and μ is the control parameter. We find T_{c1} for dDW as

$$-\ln\left(\frac{T_{c1}}{T_{c10}}\right) = Re\Psi\left(\frac{1}{2}+\frac{i\mu}{2\pi T_{c1}}\right)-\Psi\left(\frac{1}{2}\right), \qquad (17)$$

where $\Psi(z)$ is the digamma function. We note that eq. (17) is the same for both s-wave and d-wave superconductors in the Pauli limiting [87-89], where $\mu = g\mu_B H$ and μ_B the Bohr magneton.

As shown in figure. 4. the figure for T_{c1} bands back for $\mu/\Delta_{10} > 0.57$ where $\Delta_{10} \simeq 1700K$ [90]. There is no homogenous dDW beyond this point. However, we shall introduce a periodic dDW with $\Delta_1 \sim (\cos(qx)+\cos(qy))$ and $\sim (\cos(\frac{q}{2}(x+y))+\cos(\frac{q}{2}(x-y))$ and solution is given by

$$-\ln\left(\frac{T_{c1}}{T_{c10}}\right) =$$
$$Re\left\langle(1\pm\cos(2\phi))\Psi\left(\frac{1}{2}-\frac{i\mu(1-p\cos\phi)}{2\pi T_{c1}}\right)-\Psi\left(\frac{1}{2}\right)\right\rangle. \qquad (18)$$

These are analog of the Fulde-Ferrel-Larkin-Ovchinikov (FFLO) state in d-wave superconductors [90,91]. These phase boundaries are shown in figure 4[14]. Note that the hole concentration $x \sim \mu$ [93]. As to the superconducting part of the phase diagram it has not been worked out. However, in the limit $\Delta_2, \mu \ll \Delta_1$ we find

$$\lambda_2^{-1} - \lambda_1^{-1} \simeq \frac{2\mu^2}{\Delta^2(T)}\left(1-2\ln 2\frac{T}{\Delta_0}\right), \qquad (19)$$

where

$$\Delta^2(T) = \Delta_1^2(T) + \Delta_2^2(T) \simeq \Delta_1^2(0) \equiv \Delta_0^2., \qquad (20)$$

Further eq. (20) is solved as

$$\left(\frac{\Delta_2(T)}{\Delta_2(0)}\right)^2 \simeq 1 - 2\ln 2\frac{T}{\Delta_0}\left(\frac{\Delta_1(0)}{\Delta_2(0)}\right)^2, \qquad (21)$$

or

$$T_c \simeq \frac{1}{2\ln 2}\frac{\Delta_2^2(0)}{\Delta_0}. \qquad (22)$$

On the other hand the superfluid density is given by

$$\rho_s(T) = 2\pi T \Delta_2^2(T) \sum_{n\geq 0} Re\left\langle f^2 d^{-3/2}\right\rangle, \qquad (23)$$

where d has been defined in eq. (16). For $\Delta_2(T) << \Delta$ eq. (23) reduces

$$\rho_s(0) = \left(\frac{\Delta_2(0)}{\Delta_0}\right)^2 \quad \text{or} \quad T_c = \frac{1}{2\ln 2}\Delta\rho_s(0). \qquad (24)$$

Since $\lambda^{-2}(0) = (4\pi\varepsilon^2/m^*)x\rho_s(0)$, eq. (24) is interpreted as the celebrated Uemura relation[94], which can not be found in BCS theory. In other words the Uemura relation is found in the gossamer superconductivity.

4. Summary and Outlook

It is the time to reflect the principal questions running through the high T_c cuprate superconductivity in the past 20 years. a) Is the normal state Fermi liquid? b) Is the superconductivity of the BCS type? c) What is the nature of the pseudogap phase? d) Is the doped Mott-Insulator crucial? e) Is the presence of the quantum critical point (QCP) relevant? As we have already seen the renormalization group study says yes both for a) and b). Also recent study of the BCS theory of d-wave superconductivity says yes for a and b [29]. In section 4 we have shown that the pseudogap phase should be dDW. Also the low energy effective Hamiltonian appears to describe the phase diagram of the high T_c cuprate superconductivity in terms of the gossamer superconductivity, though this part of our program is still unfinished. If all these turn out to be true, it will appear that the doped Mott-Insulator and the QCP have only remote relevance. On the other hand the high T_c cuprate superconductivity appears to have a strong similarity with the superconductivity in CeCoIn$_5$ and in κ-(ET)$_2$ salts. Indeed the unconventional condensate will take center stage in the 21st century. They are a) unconventional superconductor b) unconventional density wave and c) gossamer superconductivity (i.e. unconventional superconductivity in the presence of unconventional density wave). Then the first crucial step is the identification of the symmetry of the gap functions $\Delta(k)$. Then we can construct the low energy effective Hamiltonian. These provide a useful roadmap and compass for the exploration of the new world of unconventional condensates. We are just at the beginning of this exciting game. In a vast forest inhabit plethora of unconventional condensates and wait for exploration.

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