Quantum averaging and resonances: two-level atom in a one-mode classical laser field

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Abstract

We use a nonperturbative method based on quantum averaging and an adapted from of resonant transformations to treat the resonances of the Hamiltonian of a two-level atom interacting with a one-mode classical field in Floquet formalism. We illustrate this method by *extraction of effective Hamiltonians* of the system in two regimes of weak and strong coupling. The results obtained in the strong-coupling regime, are valid in the whole range of the coupling constant for the one-photon zero-field resonance.

Keywords: quantum averaging, two-level atom, Floquet formalism, resonant transformation, nonlinear resonance

1. Introduction

Treatment of resonances in quantum systems cannot be performed by perturbative approaches. In the vicinity of resonances the perturbative formulas display small denominators that lead to the divergence of the perturbative expansions. A widely use model that incorporates a one-photon resonance is the Jaynes-Cummings Hamiltonian extracted from the full dressed Hamiltonian that describes a two-level system coupled with a single mode of a quantized field [1]. Its counterpart for an interaction with a classical laser field in the rotating-wave approximation (RWA) is the RWA Hamiltonian [2]. The RWA is valid only when an atom is perturbed by a weak and near-resonant (classical or quantized) field. For large values of the field amplitude (or coupling constant), the RWA cannot be invoked and one has to take in to account additionally the counter-rotating terms of the Hamiltonian. The goal is to obtain the spectrum for a whole interval of values of a parameter like the coupling constant beyond RWA. This is needed e.g. in applications where the coupling changes adiabatically [3], corresponding e.g. to envelopes of laser pulses or to transversal spatial profiles of cavity fields.

The semiclassical model of a two-level atom coupled with several incommensurate frequencies [4] of a classical field has been treated by different methods in Refs. [5-8]. In Ref. [9], we used the method of quantum averaging and resonant transformations proposed in [10] to detect the resonances of the model of a two-level atom interacting with a single-mode *quantized* field in two regimes of weak and strong coupling.

In this paper, we use the same method as [10] with an adapted form of resonant transformation to obtain the effective Hamiltonians of the model of a two-level atom interacting with a single-mod *classical (laser) field* in Floquet formalism. In the weak-coupling regime we have to iterate this procedure several times to derive the essential structure of the spectrum in large ranges of the coupling constant. In the strong-coupling regimes, the qualitative properties of the spectrum can be globally obtained by some preliminary unitary transformations and one resonant transformation which treat the zero-field resonances. In the case of zero-field one-photon resonance we obtain an accurate approximation, valid for all values of the coupling constant, that contains all the qualitative structure. Once this main structure is obtained, one can systematically improve the quantitative accuracy of the spectrum by applying perturbative methods.

This paper is organized as follows: In the next section, we explain the quantum averaging technique, the resonance concept, KAM-type perurbative transformations, and resonant transformation. In Sec. 3 we overview the Floquet formalism, to be self-contained, which is an essential tool to study laser-driven systems. Section 4 is devoted to the presentation of the model and some preliminary considerations. In Sec. 5, taking into account the resonances of this model in the weak-coupling regime, we extract the effective Hamiltonians by quantum averaging technique and resonant transformations. In Sec. 6 we extract the effective Hamiltonians in the strong-coupling regime. Finally, in Sec. 7 we give some conclusions and summary.

2. Resonance analysis and the method

We consider a Hamiltonian $H = H_0 + \varepsilon V$ where H_0 is the reference (unperturbed) Hamiltonian, εV is the perturbation and ε is an ordering parameter [18]. The first analysis of this problem is in terms of perturbation theory: we look for a KAM-type unitary transformation $e^{\varepsilon W_1}$ close to the identity that allows reducing the order of the perturbation from ε to ε^2:

$$e^{-\varepsilon w_1} H e^{\varepsilon w_1} = H_0 + \varepsilon D_1 + \varepsilon^2 V_2. \qquad (1)$$

εD_1 is a remaining term of order ε that satisfies $[H_0, D_0] = 0$. The unknown W_1 and D_1 are solution of the following equations (Cohomological equation) [10, 11]:

$$[H_0, W_1] + V_1 = D_1 , \qquad (2a)$$

$$[H_0, D_1] = 0 . \qquad (2b)$$

The remaining perturbation of order ε^2 is given by

$$\varepsilon^2 V_2 = \sum_{m=2}^{\infty} \frac{\varepsilon^m}{m!}((m-1)L_{W_1}^{m-1}V + L_{W_1}^{m-1}D_1) , \qquad (3)$$

where L_{W_1} is defined as:

$$L_{W_1}B = [B, W_1]. \qquad (4)$$

The solutions of eqs. (2a,b) can be written in terms of averaging [10,12-14]:

$$D_1 = \overline{V} \equiv \prod_{H_0} V \coloneqq \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} ds e^{-iH_0 S} V e^{iH_0 S}$$
$$= \sum_{v,j,j'} |v,j\rangle\langle v,j|V|v,j'\rangle\langle v,j'|, \qquad (5a)$$

$$W_1 = \lim_{x \to \infty} \frac{-i}{\tau} \int_0^{\tau} ds \int_0^{s} ds' e^{-iH_0 s'}(V - \prod_{H_0} V)e^{iH_0 s'}$$
$$= -\sum_{v\cdot j, j', v' \neq v} \frac{|v,j\rangle\langle v,j|V|v,j'\rangle\langle v',j'|}{E_v^{(0)} - E_{v'}^0}, \qquad (5b)$$

where v labels the different eigenvalues $E_v^{(0)}$ of H_0, and j is a degeneracy index which distinguishes different basis vectors $|v,j\rangle$ of the degeneracy eigenspace, and the units have been chosen such that $\hbar = 1$. the operator $\prod_{H_0}$ is the projector on the kernel of the application $A \to [H_0, A]$. We can iterate the KAM procedure by the second transformation $e^{\varepsilon^2 W_2}$ taking $H_0 + \varepsilon D_1$ as the new reference Hamiltonian and $\varepsilon^2 V_2$ as the new perturbation:

$$e^{-\varepsilon^2 W_2}(H_O + \varepsilon D_1 + \varepsilon V^2)e^{\varepsilon^2 W_2} = H_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \varepsilon^4 V_3 \qquad (6)$$

where $[D_2, H_0] = [D_2, D_1] = 0$. Notice that the new perturbation is not of order as ε^3 as would be the case in a standard perturbation theory, but is of order ε^4. similarly after n iteration we obtain:

$$e^{-\varepsilon^{2^{n-1}}W_n} \ldots e^{-\varepsilon^2 W_2}(H_0 + \varepsilon D_1 + \varepsilon^2 V_2)e^{\varepsilon^2 W_2} \ldots e^{-\varepsilon^{2^{n-1}}W_n}$$
$$= H_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots + \varepsilon^{2^{n-1}}D_n + \varepsilon^{2^n}V_{n+1} , \qquad (7)$$

where D_n, W_n, and V_{n+1} are calculated in a similar way as eqs. (5a,b) and (3) by

$$D_n = \prod_{H_{n-1}^{eff}} V_n,$$

$$W_n = \lim_{x \to \infty} \frac{-i}{\tau} \int_0^{\tau} ds \int_0^{s} ds' e^{-iH_{n-1}^{eff}s'}(V_n - \prod_{H_{n-1}^{eff}} V_n)e^{iH_{n-1}^{eff}s'},$$

$$\varepsilon^{2^n}V_{2n+1} = \sum_{m=2}^{\infty} \frac{\varepsilon^{m2^{n-1}}}{m!}((m-1)L_{W_n}^{m-1}V_n + L_{W_n}^{m-1}D_n), \qquad (8)$$

With

$$H_{n-1}^{eff} = H_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots + \varepsilon^{2^{n-2}}D_{n-1}. \qquad (9)$$

Since the perturbation term after n-th KAM iteration is order of ε^{2^n} (in contrast with the order of ε^n in the standard perturbation theory), the KAM algorithm is called superconvergent. In the following discussion, we do not write explicitly the ordering parameter ε.

A resonance is defined as a degeneracy of an eigenvalue $E_v^{(0)}$ of H_0 and is said to be *active* if the perturbation V has nonzero matrix elements in the degeneracy subspace of $E_v^{(0)} : \langle v, j|V|v, j'\rangle \neq 0$ for some j, j'. Otherwise the resonance is called passive or mute. In the case of quasi-resonance, the perturbative expansion cannot be expected to converge [9,10]. Indeed the perturbative formulas diverge close to resonances due to the appearance of *small denominators*. Using the method presented in Ref. [10], such divergences can be avoided.

The concept of resonance is defined intrinsically with respect to H_0, while the distinction between active and passive depends on the relation between H_0 and V. The analysis of resonances involves thus three aspects: (i) Decomposition of the Hamiltonian into $H = H_0 + V$.

Different decompositions can be considered for different regimes of the parameter of H. (ii) Determination of degenerate eigenvalues of the unperturbed Hamiltonian. (iii) Detection of the *resonant terms* in the perturbation V that couple these degenerate eigenstates.

The resonance terms of V can be detected by projectors of type $\prod_{H_0}$ that extract a block-diagonal part of V relative to H_0, where the blocks are generated by the degeneracy subspaces. In the absence of active resonance, when all the eigenvalues of H_0 are non-degenerate or when the resonances are mute, the matrix

representation of $\prod_{H_0}$ is in fact diagonal in the eigenbasis of H_0. in presence of active resonances, the block-diagonal effective Hamiltonian that take into account the considered resonance of the original Hamiltonian can be written as

$$H^{eff} = H_0 + \prod_{H_0} V. \tag{10}$$

We will call the transformation that diagonalizes H^{eff} *Resonant Transformation* (RT). The Hamiltonian is transformed under $\mathcal{R}$ as follows:

$$H_1 = \mathcal{R}^\dagger H \mathcal{R} = \mathcal{R}^\dagger H^{eff} \mathcal{R} + \mathcal{R}^\dagger (V - \prod_{H_0} V) \mathcal{R}$$
$$=: H_1^{(0)} + V_1, \tag{11}$$

Where $H_1^{(0)}$ is defined as the new renormalized reference Hamiltonian and V_1 is the new perturbation. The effect of $\prod_{H_0} V$ in eq. (10) is to lift the degeneracy of H_0 (completely or partially). this can happen in two ways: either the active resonance is transformed into a passive one (e.g. in the case of zero-field resonances) or the resonance disappears completely (when a crossing is transformed into an avoided crossing). The new Hamiltonian H_1 can however have other resonance at different values of the coupling parameter. If $H_1^{(0)} + V_1$ does not have any other active resonance in the considered range of the coupling constant, we can at a second stage improve the spectrum by a perturbative expansion which is expected to converge. If there are active resonances, we have to iterate the renormalization procedure by applying another RT.

3. Semiclassical Floquet formalism
In this section we overview the Floquet formalism [3,10,15] in order to provide an essential tool which will be used in the next sections. We consider the atom laser interaction in a dipole coupling approximation. When the time dependence of the periodic Hamiltonian is introduced through the time evolution of the initial phase, i.e. the amplitude and frequency of the laser field are time-independent, the Floquet formalism is just mathematically convenient tool that allows us to transform the Schrödinger equation with a time-dependent Hamiltonian into an equivalent equation with a time-independent Hamiltonian. This new equation is defined on an enlarged Hilbert space. The time dependence has been substituted by the introduction of one auxiliary dynamical variable for each laser frequency. The semiclassical Hamiltonian can be written as

$$H = H(\theta_0 + \omega t) = H_0 - \mu \varepsilon \cos(\theta_0 + \omega t), \tag{12}$$

Where μ is the dipole moment and H_0 the Hamiltonian of the free molecule, ε is the amplitude of the electric field and ω its frequency. The corresponding Schrödinger equation

$$i\hbar \frac{\partial}{\partial \theta} \Phi = H(\theta_0 + \omega t)\Phi, \qquad \Phi \in \mathcal{H} \tag{13}$$

is defined on a Hilbert space $\mathcal{H}$, which can be of infinite dimension (e.g. the space of square-integrable function $\mathcal{H} = L_2(\mathbf{R}^n, d^n x)$, where n is the number of the degrees of freedom of the molecule) or infinite dimension (e.g. in N-level models $\mathcal{H} = \mathbf{C}^N$). In eq. (13) the initial phase θ_0 appears as a parameter. One can think of eq. (13) as a family of equation parameterized by the angle θ_0. We denote the corresponding family of propagators by $U(t, t_0; \theta_0)$, which describe the time evolution of arbitrary initial condition $\Phi(t_0)$:

$$\Phi(t) = U(t, t_0; \theta_0)\Phi(t_0), \tag{14}$$

and satisfy

$$i\hbar \frac{\partial}{\partial t} U(t, t_0; \theta_0) = H(\theta_0 + \omega t)U(t, t_0; \theta_0),$$
$$U(t, t; \theta_0) = \mathbf{1}_{\mathcal{H}}. \tag{15}$$

The Floquet Hamiltonian K, also called *quasienergy operator*, is constructed as follows: we define an enlarged Hilbert space

$$\mathcal{K} := \mathcal{H} \otimes \mathcal{L}, \tag{16}$$

where $\mathcal{L} = L_2(\mathbf{S}^1, d\theta/2\pi)$ denotes the space of square integrable function on the circle $\mathbf{S}^1$ of length 2π, with a scalar product

$$\langle f_1 | f_2 \rangle_{\mathcal{L}} = \int_{\mathbf{S}^1} \frac{d\theta}{2\pi} f_1^*(\theta) f_2(\theta). \tag{17}$$

This space is generated by the orthonormal basis $\{e^{ik\theta}, k \in z\}$.

We first *lift* the family of operators $U(t, t_0; \theta_0)$ (defined on $\mathcal{H}$) into an operator $U(t, t_0)$ acting on the enlarged space $\mathcal{K}$ by considering the initial phase θ_0 as a multiplication operator θ. This operator is unitary in $\mathcal{K}$. On the enlarged Hilbert space $\mathcal{K}$ the Floquet Hamiltonian K is defined as :

$$K = -i\hbar\omega \frac{\partial}{\partial \theta} + H. \tag{18}$$

In this expression H is just the semiclassical Hamiltonian (12) but with the phase taken as a dynamical variable. The usefulness of Floquet Hamiltonian comes from the fact that is time-independent and that the dynamics it defines on $\mathcal{K}$ is essentially equivalent with the one of eq. (13). The Floquet Hamiltonian K defines a time evolution in $\mathcal{K}$ through the equation

$$i\hbar \frac{\partial}{\partial t} \Psi = K\Psi, \qquad \Psi \in \mathcal{K} = \mathcal{H} \otimes \mathcal{L}. \tag{19}$$

This time evolution can be expressed in terms of propagator U_K characterized by

$$i\hbar \frac{\partial}{\partial t} U_K(t,t_0) = K U_K(t,t_0), \quad U_K(t,t) = \mathbf{1}_\mathcal{K}, \tag{20}$$

i.e. $\Psi(t) = U_K(t,t_0)\Psi(t_0)$, (where $\mathbf{1}_\mathcal{K}$ is the identity operator in $\mathcal{K}$). Since K is time-independent, the propagator can be written as

$$U_K(t,t_0) = U_K(t - t_0) = e^{-iK(t-t_0)/\hbar}. \tag{21}$$

In order to establish a relation between U and U_K we define the following phase translation operator $\mathcal{T}_{\omega t}$, which acts on $f \in \mathcal{L}$, by

$$\mathcal{T}_{\omega t} = f(\theta) := f(\theta + \omega t) \tag{22}$$

And can be expressed as

$$\mathcal{T}_{\omega t} = e^{\omega t \frac{\partial}{\partial \theta}}. \tag{23}$$

The relation between U and U_K cab be expressed by

$$\mathcal{T}_{-\omega t} U \mathcal{T}_{\omega t_0} = U_K(t - t_0) \equiv e^{-ik(t-t_0)/\hbar} \tag{24}$$

It implies that if is a solution of eq. (19) then we can obtain a solution of eq. (13) by

$$\Phi(t) = [\mathcal{T}_{\omega t} \Psi(t)]_{\theta = \theta_0} = \Psi(t, \theta_0 + \omega t), \tag{25}$$

where fixing a constant value for the dynamical variable means returning into the original Hilbert space $\mathcal{H}$ [15].

4. Two - level atom in a one- mode classical field

In this section, we consider the model of a two-level atom coupled with a single-frequency laser field. The Floquet Hamiltonian acting on the enlarged Hilbert space $\mathcal{K} = \mathcal{L} \otimes \mathcal{H}, \mathcal{H} = \mathbf{C}^2$, is given by

$$K = -i\omega \frac{\partial}{\partial \theta} \otimes \mathbf{1}_2 + \mathbf{1}_\mathcal{L} \otimes \frac{\omega_0}{2} \sigma_z + \Omega \cos\theta \otimes \sigma_z, \tag{26}$$

where $\Omega = -\mu\varepsilon$ is the coupling of the atomic transition coupled by laser field, and σ_z, σ_x are Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{27}$$

where $\mathbf{1}_2$ is the 2×2 identity matrix. This Hamiltonian is identical with the Hamiltonian of a two-level atom in a one-mode *quantized* field,

$$H = \omega a^\dagger a \otimes \mathbf{1}_2 + \mathbf{1} \otimes \frac{\omega_0}{2} \sigma_z + g(a + a^\dagger) \otimes \sigma_x, \tag{28}$$

in the limit of *infinite cavity volume, large photon number average, and constant photon density* [15]. Since $\cos\theta = (e^{-i\theta} + e^{i\theta})/2$, comparing eqs. (26) and (28), we can establish a correspondence among the operators $a, a^\dagger$ and $e^{i\theta}, e^{-i\theta}$. This point will be clarified further in the next section.

The parity operator of this system is

$$P = e^{+\pi \frac{\partial}{\partial \theta}} \otimes \sigma_z, \tag{29}$$

with the properties

$$[P,K] = 0, \quad P = P^\dagger, \quad P^2 = \mathbf{1}_\mathcal{K} \equiv \mathbf{1}_\mathcal{L} \otimes \mathbf{1}_2. \tag{30}$$

As a consequence, the eigenstsate of K can be separated in two symmetry classes, even or odd, under P:

$$P|\phi_k, \pm\rangle = \pm|\phi_k, \pm\rangle, \quad K|\phi_k, \pm\rangle = E_{k,\pm}|\phi_k, \pm\rangle. \tag{31}$$

The conceptual framework for the solution of the solution of this system based on the construction of unitary transformation can be described as follows: First, we decompose the Hamiltonian in two terms as $H = H_0 + V$. Depending on the considered range of the parameters of the system, different decomposition may be considered. H_0 is *a priori* an operator that is a regular function exclusively of the operators $N := -i\frac{\partial}{\partial \theta}$ (photon-number operator) and σ_z . The operators N and σ_z can be considered in the present model as quantum analogues of classical global actions [16], and H_0 can be labelled *integrable*. The perturbation V contains functions that involve also the other operators $e^{-i\theta}, e^{i\theta}, \sigma_x, \sigma_y$. The goal is to determine a unitary transformation U, that should be expressed in terms of well-behaved regular functions of $e^{-i\theta}, e^{i\theta}, \sigma_x, \sigma_y, \sigma_z$,such that:

$$U^\dagger (H_0(N, \sigma_Z) + V(e^{-i\theta}, e^{i\theta}, \sigma_x, \sigma_y, \sigma_z))U = H'(N, \sigma_z), \tag{32}$$

where H' is a regular function f exclusively of the action operators $N, \sigma_z : H'(N, \sigma_Z) = f(n, \sigma_z)$. With this transformation the eigenvectors of H can be expressed as $|\phi_K, \pm\rangle = U(|k\rangle \otimes |\pm\rangle)$ and the corresponding eigenvalues as $E_{k,\pm} = f(k, \pm 1)$ where $N|k\rangle = k|k\rangle$ and $\sigma_z|\pm\rangle = \pm|\pm\rangle$.

Most of the perturbative approaches can be interpreted as methods to find approximation of the transformation of U. The presence of resonance is one of the central difficulties in the construction of U, as will be made precise below. In this paper we discuss an iterative approach that consist of constructing first some approximation of U that take into account the dominating effects of certain number of resonances. The transformations involved in this stage are far from the identity and have a clearly non-perturbative character. Once we have a transformation that takes into account the main effect of a set of resonances that are relevant in a considered interval of coupling constant Ω, a perturbative approach can be applied to improve the approximation quantitatively. The transformations involved in this second stage can be considered as deformations of the identity, since they can be written in the form e^W . This stage cannot be implemented if the resonances are not taken care of beforehand.

As in classical mechanics, the construction U leading to Hamiltonian contains only action variables can often be considered in two steps: $U = U_1 U_2$. In the first step, that is called *reduction*, the Hamiltonian is transformed by into a form that contains functions of $\sigma_z, \sigma_x, \sigma_y$ and N, but not of $e^{-i\theta}$ and $e^{i\theta}$. The degree of freedom of the field is made trivial and the number of non-trivial degrees of freedom is thus reduced by one. When we apply this reduction to the effective Hamiltonian, we obtain a *reduced effective Hamiltonian*. We remark that in the literature, this reduced effective Hamiltonian is often called simply effective Hamiltonian.

In the second step, the reduced Hamiltonian is transformed under U_2 into a form that contains functions of only N and σ_z. For the model (26), the reduction step corresponds to a diagonalization in the space $\mathcal{L}$ and the second step corresponds to a diagonalization in the atomic Hilbert space which in this case trivial. The construction of the RT is based on this reduction procedure.

5. Effective Hamiltonians in the weak - coupling regime

The Floquet Hamiltonian (26) in the weak-coupling regime can be decomposed as follows:

$$K_0 = -i\omega \frac{\partial}{\partial \theta} + \frac{\omega_0}{2}\sigma_z, \qquad V = \Omega \cos\theta \, \sigma_x . \tag{33}$$

The eigenvalues and eigenvectors of K_0 are

$$E_{k,\pm}^{(0)} = k\omega \pm \omega_0/2,$$

$$\phi_{k,\pm}^{(0)} = |k,\pm\rangle, \quad |k,+\rangle = \begin{pmatrix} |k\rangle \\ 0 \end{pmatrix}, \quad |k,-\rangle = \begin{pmatrix} 0 \\ |k\rangle \end{pmatrix}, \tag{34}$$

where $|k\rangle$ is eigenvector of the relative photon number operator $-i\frac{\partial}{\partial \theta}$ as $-i\frac{\partial}{\partial \theta}|k\rangle = k|k\rangle$ with orthonormal condition

$$\langle k||k'\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i(k-k')\theta} = \sigma_{k,k'} . \tag{35}$$

For the case of one-photon resonance $\omega = \omega_0$, the eigenvalues of K_0 are degenerate as $E_{k,+}^{(0)} = E_{k+1,-}^{(0)}$. The degeneracy eigenspaces are spanned by vectors $\phi_{k,+}^{(0)}$ and $\phi_{k+1,-}^{(0)}$. The resonant part of V is obtained by eq. (5a):

$$V_{res} := \Pi_{K_0} V = \sum_{K=-\infty}^{\infty} (|k,+\rangle\langle k,+|V|K+1,-\rangle\langle K+1,-|$$

$$+ |K+1,-\rangle\langle K+1,-|V|k,+\rangle\langle k,+|)$$

$$= \Omega/2 \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}, \tag{36}$$

where we have obtained the resonant elements of V as follows:

$$\langle k+1,-|V|k,+\rangle = \Omega\langle k+1|\cos\theta|k\rangle\langle -|\sigma_x|+\rangle$$

$$= \Omega\langle k+1|\cos\theta|k\rangle$$

$$= \Omega\int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i(k+1)\theta} \cos\theta \, e^{ik\theta}$$

$$= \Omega/2 , \tag{37}$$

and

$$\sum_k |k+1,-\rangle\langle k,+| = e^{i\theta} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{38}$$

The spectral representation of the multiplication operator $e^{i\theta}$ and the creation operator $a^\dagger$ can be compared as follows:

$$e^{i\theta} = \sum_{k=-\infty}^{\infty} |k+1\rangle\langle k| \leftrightarrow a^\dagger = \sum_{n=-\infty}^{\infty} \sqrt{n+1}|n+1\rangle\langle n| \tag{39}$$

Therefore, the effective Floquet Hamiltonian containing the one-photon resonance (corresponding to the Jaynes-Cummings Hamiltonian [1]) is

$$K_0^{eff} = -i\omega \frac{\partial}{\partial \theta} + \frac{\omega}{2}\sigma_z + \Omega/2 \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}. \tag{40}$$

Next we transform K_0^{eff} by a RT to a regular function of exclusively the action operators $-i\omega\frac{\partial}{\partial \theta}, \sigma_z$. The reduction step of RT denoted R is a unitary (as opposed to isometric in the quantized field [9]) transformation that can eliminate the θ- dependence in the resonant term,

$$R^\dagger V_{res} R = \frac{\Omega}{2}\sigma_x. \tag{41}$$

The suitable transformation satisfying this condition is

$$R := \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} \sum_{k=\infty}^{\infty} |k\rangle\langle k+1| & 0 \\ 0 & 1 \end{pmatrix}. \tag{42}$$

Applying R on K gives:

$$R^\dagger K R = -i\omega \frac{\partial}{\partial \theta} - \frac{\omega}{2} + \frac{\Omega}{2}\sigma_x + \frac{\Omega}{2} \begin{pmatrix} 0 & e^{2i\theta} \\ e^{-2i\theta} & 0 \end{pmatrix} \tag{43}$$

where we have used the relation

$$\frac{\partial}{\partial \theta} R = R\frac{\partial}{\partial \theta} + \frac{\partial R}{\partial \theta} . \tag{44}$$

We remark that when there is a small detuning, $\omega_0 = \omega + \Delta$, we can decompose K as $K_0 = -i\omega\frac{\partial}{\partial \theta} + \frac{\omega}{2}\sigma_z$ and $V = \frac{\Delta}{2}\sigma_Z + \Omega\cos\theta\,\sigma_x$, where the detuning has been included in the perturbation. This decomposition leads to the same resonant term as eq. (36), and thus the eq. (43) is modified by adding the term

$\frac{\Delta}{2}\sigma_z$. The corresponding reduced effective Hamiltonian in the atomic Hilbert space and in the rotating-wave approximation is

$$H_W^{eff} = \begin{pmatrix} \frac{\Delta}{2} & \frac{\Omega}{2} \\ \frac{\Omega}{2} & -\frac{\Delta}{2} \end{pmatrix}. \tag{45}$$

We note that the Rabi frequency of a system having a reduced effective Hamiltonian such as eq. (45) is

$$r = \Omega/2. \tag{46}$$

This means that if the system starts from the state $|+\rangle$ or $|-\rangle$, the system will populate periodically at later times the states $|+\rangle$ and $|-\rangle$ with period of $2\pi/\omega$. On the other hand, we will see in section VI that the Rabi frequency of this system in the strong-coupling regime is different from eq. (46).

The second step of the first RT consists in diagonalizing the σ_x term in $R^\dagger K_0^{eff} R$ by a $\pi/2$ rotation around the y-axis,

$$T = e^{-i\frac{\pi}{4}\sigma_y} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{47}$$

with the properties

$$T^\dagger \sigma_X T = \sigma_Z, \quad T^\dagger \sigma_Z T = -\sigma_X, \tag{48}$$

as follows:

$$K_1 = T^\dagger R^\dagger KRT = -i\omega\frac{\partial}{\partial\theta} - \frac{\omega}{2} + \frac{\Omega}{2}\sigma_x + \frac{\Omega}{2}\begin{pmatrix} \cos 2\theta & i\sin 2\theta \\ -i\sin 2\theta & \cos 2\theta \end{pmatrix}. \tag{49}$$

Now, we can decompose the Hamiltonian K_1 to:

$$K_0 = -i\omega\frac{\partial}{\partial\theta} - \frac{\omega}{2} + \frac{\Omega}{2}\sigma_z, \quad V_1 = \frac{\Omega}{2}\begin{pmatrix} \cos 2\theta & i\sin 2\theta \\ -i\sin 2\theta & \cos 2\theta \end{pmatrix}, \tag{50}$$

and the resonant term of V_1 is extracted by $\prod_{K_1^{(0)}} V_1$ in terms of the degeneracies of $K_1^{(0)}$. The resonances of K_1 are nonlinear as the degeneracies of $K_1^{(0)}$ are Ω-dependent. This procedure can be iterated to obtain effective Hamiltonians for larger values of the coupling constant Ω [9,10].

6. Effective Hamiltonians in the strong - coupling regime

In this section we consider the Floquet Hamiltonian (26) in the strong-coupling regime $\Omega \gg \omega_0$ and we extend the ideas given in Refs. [9, 10] to the case of multi-photon resonances in Floquet formalism. The strong-coupling regime suggests an alternative decomposition as

$$K_0 = -i\omega\frac{\partial}{\partial\theta} + \Omega\cos\theta\sigma_x, \quad V = \frac{\omega_0}{2}\sigma_Z \tag{51}$$

where K_0 contains again all the unbounded operators of the system, and the perturbation V is a bounded operator. We take an approach which is very similar to the one of Ref. [9]. In the case of one-photon resonance $\omega_0 = \omega$, this approach allows us to obtain an effective Hamiltonian which describes the essential structure of the system for the whole range of values of the coupling $\Omega \in [0,\infty)$. However for multi-photon resonances $\omega_0 = m\omega$, $m = 3,5,7,\dots$, this approach works only for large values of Ω. As in [9], we first apply the transformation (47) to diagonalize K_0 in the atomic Hilbert space

$$T^\dagger KT = -i\omega\frac{\partial}{\partial\theta} + \Omega\cos\theta\sigma_z - \frac{\omega_0}{2}\sigma_x. \tag{52}$$

Next we apply a second unitary transformation to eliminate the θ-dependence in $T^\dagger K_0 T$:

$$U = \begin{pmatrix} e^{-i\frac{\Omega}{\omega}\sin\theta} & 0 \\ 0 & e^{i\frac{\Omega}{\omega}\sin\theta} \end{pmatrix}, \tag{53}$$

which leads to

$$U^\dagger T^\dagger KTU = -i\omega\frac{\partial}{\partial\theta} - \frac{\omega_0}{2}\begin{pmatrix} 0 & e^{+i\gamma\sin\theta} \\ e^{-i\gamma\sin\theta} & 0 \end{pmatrix}, \tag{54}$$

where

$$\gamma := \frac{2\Omega}{\omega}. \tag{55}$$

Now by using the relation

$$e^{\pm i\gamma\sin\theta} = \cos(\gamma\sin\theta) \pm i\sin(\gamma\sin\theta)$$
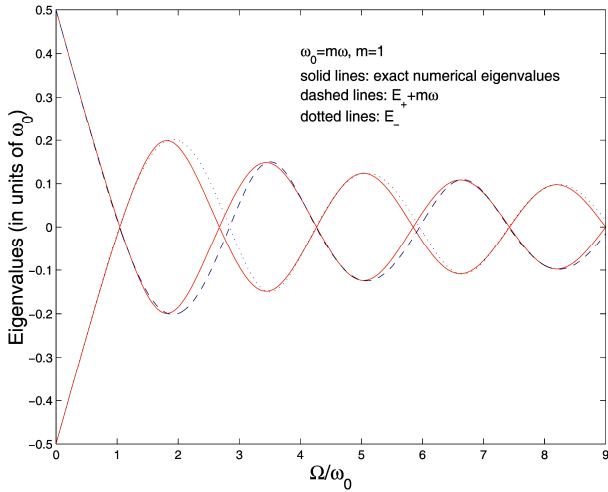$$= J_0(\gamma) + 2\sum_{p=2,4,\dots} J_P(\gamma)\cos p\theta \pm 2i\sum_{q=1,3,\dots} J_q(\gamma)\sin q\theta \tag{56}$$

we can write $U^\dagger T^\dagger KTU$ in terms of harmonics of the new perturbation as

$$U^\dagger T^\dagger KTU = -i\omega\frac{\partial}{\partial\theta} - \frac{\omega_0}{2}J_0(\gamma)\sigma_X - \omega_0\sum_{p=2,4,\dots} J_P(\gamma)\cos p\theta\sigma_x + \omega_0\sum_{q=1,3,\dots} J_q(\gamma)\sin q\theta\sigma_y. \tag{57}$$

To diagonalize the σ_x terms, we apply the transformation T again.

$$K_1 = U^\dagger T^\dagger KTU = -i\omega\frac{\partial}{\partial\theta} - \frac{\omega_0}{2}J_0(\gamma)\sigma_Z - \omega_0\sum_{p=2,4,\dots} J_P(\gamma)\cos p\theta\sigma_Z + \omega_0\sum_{q=1,3,\dots} J_q(\gamma)\sin q\theta\sigma_y. \tag{58}$$

At a second stage, we decompose $K_1 = K_1^{(0)} + V_1$ as

Figure 1. Comparison of the exact numerical eigenvalues (solid lines) of eq. (26) as a function of the coupling constant in the resonant case $\omega_0 = \omega$, with the approximate eigenvalues (dashed and dotted lines) obtained from eq. (69).

Figure 2. Comparison of the exact numerical eigenvalues (solid lines) of eq. (26) as a function of the coupling constant in the resonant case $\omega_0 = 3\omega$, with the approximate eigenvalues (dashed and dotted lines) obtained from eq. (69).

$$K^{(0)}{}_1 = -i\omega \frac{\partial}{\partial\theta} - \frac{\omega_0}{2} J_0(\gamma)\sigma_Z,$$

$$V_1 = -\omega_0 \sum_{p=2,4,\ldots} J_P(\gamma)\cos p\theta\,\sigma_Z + \omega_0 \sum_{q=1,3,\ldots} J_q(\gamma)\sin q\theta\,\sigma_y.$$
(59)

For

$$\omega_0 J_0(\gamma) = m\omega,$$
(60)

The eigenvalues of $K_1^{(0)}$ are with degeneracies $\lambda_{k,+} = \lambda_{k-m,-}$. The eq. (60) is the *nonlinear resonance condition* as it depends on the coupling Ω. For $\Omega \to 0$ this condition is reduced to the *m*-photon zero-field resonance condition $\omega_0 = m\omega$. The resonant part of V_1 can be calculated as follows:
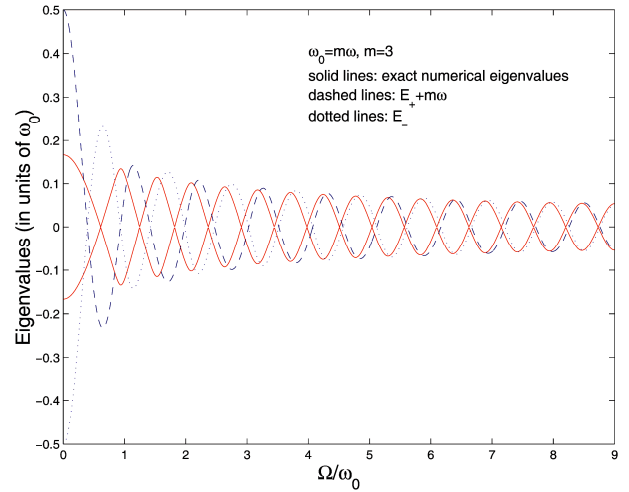
$$\Pi_{K(0)} V_1 = \sum_{k=-\infty}^{\infty} (|k-m,-\rangle\langle k-m,-|V|k,+\rangle\langle k,+|$$
$$+ |k,+\rangle\langle k,+|V|k-m,-\rangle\langle k-m,-|)$$
$$= -\frac{\omega_0}{2} J_m(\gamma)\begin{pmatrix} 0 & e^{im\theta} \\ e^{-im\theta} & 0 \end{pmatrix}, \quad m=1,3,5,\ldots$$
(61)

where we have used the relations

$$\sum_{k=-\infty}^{\infty} |k,m-1\rangle\langle k| = e^{-im\theta}, \quad \sum_{k=-\infty}^{\infty} |k\rangle\langle k,m-1| = e^{+im\theta}.$$
(62)

We observe that the terms with even *m* are absent in the average of V_1 with respect to $K_1^{(0)}$. So the effective Floquet Hamiltonian of the system in strong-coupling regime can be written as

$$K_1^{eff} = -i\omega\frac{\partial}{\partial\theta} + \frac{\omega_0}{2} J_0(\gamma)\sigma_z -$$
$$\frac{\omega_0}{2} J_m(\gamma)\begin{pmatrix} 0 & e^{+im\theta} \\ e^{-im\theta} & 0 \end{pmatrix}, \quad m=1,3,5,\ldots$$
(63)

The reduction step of the m-photon RT to eliminate the θ-dependence of the resonant term in K_1^{eff} can be performed by

$$R_m = \begin{pmatrix} 1 & 0 \\ 0 & e^{-im\theta} \end{pmatrix},$$
(64)

which leads to

$$R_m^\dagger K_1^{eff} R_m = -i\omega\frac{\partial}{\partial\theta} + \begin{pmatrix} 0 & 0 \\ 0 & -m\omega \end{pmatrix} -$$
$$\frac{\omega_0}{2} J_0(\gamma)\sigma_z - \frac{\omega_0}{2} J_m(\gamma)\sigma_x.$$
(65)

This gives the reduced effective Hamiltonian of the system, in the atomic Hilbert space, for strong-coupling regime:

$$H_S^{eff} = \begin{pmatrix} -\dfrac{\omega_0}{2} J_0(\gamma) & -\dfrac{\omega_0}{2} J_m(\gamma) \\ -\dfrac{\omega_0}{2} J_m(\gamma) & -m\omega + \dfrac{\omega_0}{2} J_0(\gamma) \end{pmatrix}.$$
(66)

Compare this Hamiltonian with eq. (45). In this case the Rabi frequency of the system, instead of $\Omega/2$, is

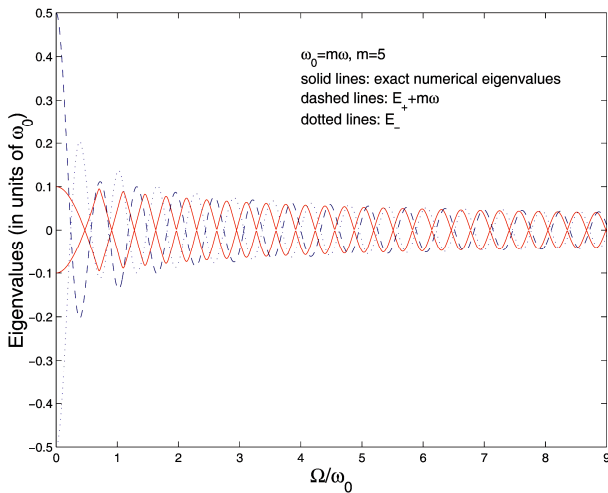$$r = \frac{\omega_0}{2} J_m(\frac{2\Omega}{\omega}).$$
(67)

One can see that for $\Omega \to 0$ and *m* = 1, the Hamiltonian (45) with zero detuning can be recovered as

$$J_0(\gamma) \to 1, \quad J_m(\gamma) \to (\frac{\gamma}{2})^m, \text{ for } \gamma \to 0,$$
(68)

The eigenvalues of eq. (66) can be calculated easily as

$$E_\pm = \frac{-m\omega}{2} \pm \frac{1}{2}\sqrt{[m\omega - \omega_0 J_0(\gamma)]^2 + [\omega_0 J_m(\gamma)]^2},$$
$$m = 1,3,5,\ldots$$
(69)

Figures 1, 2 and 3 compare the exact eigenvalues of the

Figure 3. Comparison of the exact numerical eigenvalues (solid lines) of eq. (26) as a function of the coupling constant in the resonant case $\omega_0 = 5\omega$, with the approximate eigenvalues (dashed and dotted lines) obtained from eq. (69).

system with the approximate eigenvalues obtained from eq. (69). We observe in figure 1 that for one-photon resonance the approximate eigenvalues are valid for the whole range of the coupling Ω, while for the three- and five-photon resonances in figures 2, 3 the eigenvalues obtained from eq. (69) are valid only for large values of Ω.

7. Conclusions and discussions

We have used a non-perturbative method based on the quantum averaging technique to determine the spectral properties of a semiclassical system (a two-level atom in a one-mode classical field) in two regimes of weak- and strong-coupling. It consists in

the construction of *unitary* transformations that leads to an effective reduced Hamiltonian. These transformations are adapted to the structure of the resonances. Their role is to construct a first effective Hamiltonian that contains the main qualitative features of the spectrum in a given range of the coupling parameter. The diagonalized form of this effective Hamiltonian, which depends parametrically on the coupling constant, is then taken as a new reference Hamiltonian around which one can apply perturbative techniques to improve the quantitative accuracy of the spectrum.

We have analyzed the resonances in two regimes of weak and strong coupling. The analysis of the strong-coupling regime of this model leads to results that are valid for all values of the coupling and for all energies in the case of a one-photon resonance, $\omega_0 = \omega$ when this later resonance is additionally treated. The possibility to obtain such a global result is due to a particular property of the model, and one cannot expect to obtain it for general models. The particular property is that the part we selected as the reference Hamiltonian H_0 in the strong-coupling regime contains all the unbounded operators of the complete model and is explicitly solvable. The term that was left to be treated by RT and perturbation theory is a bounded operator.

The extension of the strong-coupling regime for multi-photon resonances $\omega_0 = m\omega$, $m = 3,5,7\ldots$ to the weak-coupling regime is under consideration. This can be related to the fact that in the weak-coupling decomposition of the complete Hamiltonian (26), the multi-photon resonances are passive. These resonances could be treated after applying a KAM-type transformation [10, 17] to transform them from passive to active resonances.

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18. The parameter ε can be thought as a small parameter that can be used to attribute an order of perturbation to the terms appearing in each iteration of the KAM-type perturbation theory.