IRANIAN JOURNAL OF PHYSICS RESEARCH

Iranian Journal of Physics Research, Vol. 5, No. 1, 2005

The phase structure of two dimensional pure U(N) lattice gauge theories with complex action

B Mirza

Department of Physics, Isfahan University of Technology, Isfahan 84154-83111, Iran
E-mail: b.mirza@cc.iut.ac.ir

(Received 6 January 2004. ; in final form 9 June 2004)

Abstract

We study the phase structure of two dimensional pure lattice gauge theory with a Chern term. The symmetry groups are non-Abelian, finite and disconnected sub-groups of *SU(3)*. Since the action is imaginary it introduces a rich phase structure compared to the originally trivial two dimensional pure gauge theory. The Z_3 group is the center of these groups and the result shows that if we use one dimensional irreducible representations (irreps) for group elements the phase diagrams are similar to diagrams of Z_3 group. Other irreps with different dimensionality show a little different behaviour for the phase diagram. The phase transition for the Z_3 group is first order. The phase structure of the U(N) model is considered and it is proved that it has an infinite number of first order phase transitions.

Keywords: lattice, gauge theory

1. Introduction

We study 1+1 dimensional pure gauge theory plus a Chern term. In two dimensions any pure gauge theory is locally trivial and has no propagating modes. These models are analytically solvable and they exhibit no phase transition. However the triviality of two dimensional theories is not guaranteed for generalized actions [1-5].

These models can possess a rich phase structure if the conventional real action is replaced by a complex one[5]. These kind of actions arise from effective pure gauge models [6]. This work is a generalization of the results which has already been obtained in [5] to the U(N) and some non-Abelian, finite and disconnected sub-groups of SU(3). We shall apply the group character expansion method to calculate the partition function for some non-Abelian and finite sub-groups of SU(3). In this part we review the formulation of lattice gauge theory on a two dimensional surface without boundary[7]. On the lattice the partition function takes the form,

$$Z = \int \prod_i dU_i e^S \quad , \tag{1}$$

$$S = \frac{\beta}{N} \sum_p (trU_p + trU_p{}^{\dagger}) \quad , \tag{2}$$

where unitary N×N matrices U_i are attached to the links of the lattice. It is a consequence of the Peter-Weyl theorem that the space of class function on a compact Lie group G, is spanned by its irreducible characters i.e. by the traces in the unitary irreps $r \in G$. Since $e^{S(u)}$ is a conjugate class function on G, it can be expanded in terms of irreducible characters of G. Actually the partition function of this theory can be calculated exactly by the group character expansion,

$$e^S = \sum d_r \Lambda_r(\beta) \chi_r(U) \quad . \tag{3}$$

The sum in (eq.3) runs over all irreps r of the group. $\chi_r(U)$ is the character of r and $d_r = \chi$ (I) is its dimension. The coefficients $\Lambda_r(\beta)$ can be calculated by the following integral,

$$\Lambda_r(\beta) = \frac{1}{d_r} \int_G dU e^{S(U)} \chi_r^*(U) \quad . \tag{4}$$

In two dimensions the basic operation in calculating the partition function is gluing plaquettes along a common link. This operation is trivial because of the triviality of the orthogonality condition for characters,

$$\int dU e^{S(U_1 U) + S U^{\dagger} U_2)} = \sum_{r_1, r_2} dr_1 dr_2 \Lambda_{r_1}(\beta) \Lambda_{r_2}(\beta) \int_G Xr_1(U_1 U) Xr_2(U^{\dagger} U_2) dU \quad . \tag{5}$$

And by using the properties of characters we get to,(see figure 1),

Dedicated to the Memory of Ian Kogan

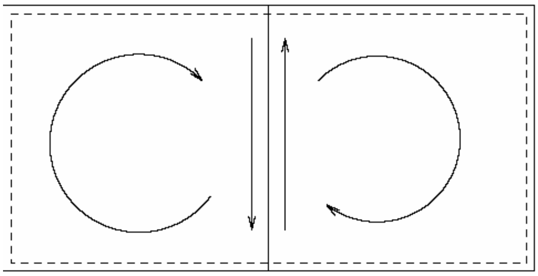

Figure 1. Making one plaquette by gluing two neighbors. Two dimensional Yang-Mills theory is a one plquette model.

$$\sum_r d_r \Lambda_r^2(\beta)\chi_r(U_1U_2) \quad . \tag{6}$$

Integrating over all links in a region with fixed boundary conditions,

$$Z_N(\beta,\Gamma) = \sum_r d_r \Lambda_r^N(\beta)\chi_r(\Gamma) \quad , \tag{7}$$

where Γ is the product of link variables around the boundary and N is the number of plaquettes contained in the domain.

As is known, one can build an arbitrary two-dimensional manifold by gluing any number of handles, orientable sheets and Mobius sheets to the sphere with holes. We are able to cover the surface of any two dimensional manifold by plaquettes and then calculate the partition function using the same procedure as for gluing plaquettes. For example for the cylinder (sphere with two holes) we have,

$$Z_{N_1N_2} = \sum_{r_1 r_2} d_{r_1} d_{r_2} \Lambda_{r_1}^{N_1}\Lambda_{r_2}^{N_2}$$

$$\int dU_1 dU_2 \chi_r(U_1W_1U_2V_1)\chi_{r_2}(V_2U_2^\dagger W_2U_1^\dagger)$$

$$= \sum_r \Lambda_r^{(N_1+N_2)}\chi_r(V_1V_2)\chi_r(W_1W_2) \quad , \tag{8}$$

where V_1V_2 and W_1W_2 are products of matrices along the boundaries of holes (figure 2) and we used the following formula,

$$d_r \int dW \chi_r(AWBW^\dagger) = \chi_r(A)\chi_r(B) \quad . \tag{9}$$

Each hole decreases the degree of d_r by one. The result for arbitrary surface with h handles (h holes) and without boundary is:

$$Z_N(\beta) = \sum_r d_r^\chi \Lambda_r^N(\beta) \quad , \tag{10}$$

where χ=2-2h is the Euler characteristic of the surface. In the thermodynamic limit (N $\to\infty$) only the term with larger Λ survives and the free energy is given by:

$$f(\beta) = -\frac{1}{\beta}\log \Lambda_x(\beta) \quad . \tag{11}$$

The string tension is defined by:

$$\sigma_q = \lim_{A\to\infty} \lim_{N\to\infty} -\frac{\log < W_q(C) >}{A} \quad , \tag{12}$$

where

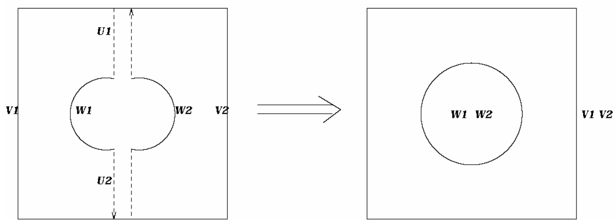

Figure 2. Cutting of a surface into parts without holes and gluing them along the common links U_1 and U_2.

$$< W_q(C) >= \frac{1}{Z_N}\sum_{\{U\}} \chi_q(\prod_C U_k) e^{\beta S\{U\}} \quad , \tag{13}$$

is the Wilson loop average of the loop C in the q representation of the gauge group and A is the area enclosed by the loop. The loop average $<W_q(C)>$ can be obtained in the same way as as the partition function, and the Wilson loop average has the following form:

$$< W_q(C) >= \lim_{N\to\infty} \frac{\sum_{p,r} d_r^{x-1} d_p \Lambda_r^{N-A}(\beta)\Lambda_p^A(\beta) D_{pq}^r}{\sum_i d_i^\eta \Lambda_i^N(\beta)}$$

$$= \sum_p \frac{d_p}{d_x}\left(\frac{\Lambda_p(\beta)}{\Lambda_x(\beta)}\right)^A D_{pd}^r \quad , \tag{14}$$

where

$$D_{pq}^r = \int dU \chi_p(U)\chi_q(U) X_r(U^\dagger) \tag{15}$$

is a factor which depends on the chosen representation of G on the two sides of the loop C. The dominating term for large loops (A $\to\infty$) will be the one with the largest Λ. The string tension is given by

$$\sigma_q(\beta) = \log\left[\frac{\Lambda_x(\beta)}{\Lambda_p(\beta)}\right] \quad . \tag{16}$$

If the largest Λ is not unique the system undergoes a phase transition caused by the swap over between the largest Λ's. In the next sections we will consider the phase structure of some non-Abelian sub groups of SU(3).

2. Phase structre of generalized action

In a continuum theory of gauge fields and fermions an effective action can be obtained by integrating the fermions. In the one loop order this action contains a Chern term[5]. It is hoped that some qualitative idea about the real four dimensional gauge theory will be found by a study of this two dimensional theory. The action consists of two parts; a real one (Wilson action) and an imaginary one (Chern term)

$$S(U) = S_1(U) + \varepsilon S_2(U) \quad , \tag{17}$$

where,

$$S_1(U) = \frac{\beta}{N}(trU + trU^\dagger) \quad , \tag{18}$$

is the usual Wilson action for pure Yang-Mills theory, and

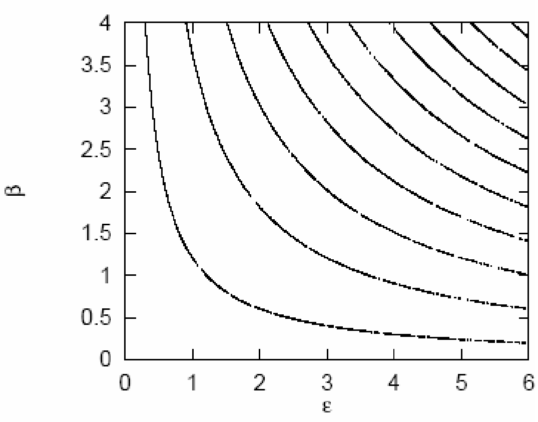

Figure 3. The phase diagram for the Z_3 group.

$$S_2(U) = \frac{\beta}{N}(trU - trU^{\dagger}) \quad , \tag{19}$$

is the Chern term and εis an additional parameter. In the ε=0 limit the action describes a pure two dimensional gauge theory. The two dimensional Yang-Mills theory is a trivial theory and the action is real and we have

$$\begin{aligned} \Lambda_r(\beta) &= \frac{1}{d_r}\int dU e^{S(U)} \chi r(U) \\ &\le \frac{1}{d_r}\int dU e^{S(U)} \chi r(U) \\ &\le \int_G dU e^{S(U)} \\ &= \Lambda_0(\beta) \quad . \end{aligned} \tag{20}$$

So Λ_0 is always larger than all the other Λ's and indeed there is no phase transition. But if we add a complex term to the action the theory is not trivial. It is possible that for some values of β and ε two different Λ_r are equal. This makes a discontinuity in the first derivative of the free energy and so a first order phase transition. For example if the gauge group is U(1) and ε=1 then the character expansion coefficients take the following simple form:

$$\begin{aligned} \Lambda_r(\beta,\varepsilon=1) &= \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{\beta e^{i\varphi}} e^{-ir\varphi} \quad , \\ \Lambda_r(\beta,\varepsilon=1) &= \frac{\beta^r}{r!} \qquad r \ge 0 \quad , \\ \Lambda_r(\beta,\varepsilon=1) &= 0 \qquad r < 0 \quad . \end{aligned} \tag{21}$$

At β_r=r+1 a swap over between Λ_r and Λ_{r+1} takes place which causes a first order phase transition. In general the same thing happens for the U(N) group. Consider an element of U(N) which is represented by a N×N unitary matrix and the irreducible representations of U(N) are labeled by a set of N positive or negative integers [4]:

$$\{\lambda\} = \{\lambda_1 \ge \lambda_2 ... \ge \lambda_N\}(\lambda_1 \ge 0 \;\; or < \lambda_N) \quad , \tag{22}$$

or alternatively if $l_i = \lambda_i + N - i$

$$l_1 > l_2 > ... > l_N . \tag{23}$$

After the same calculation and integral over the compact group one has:

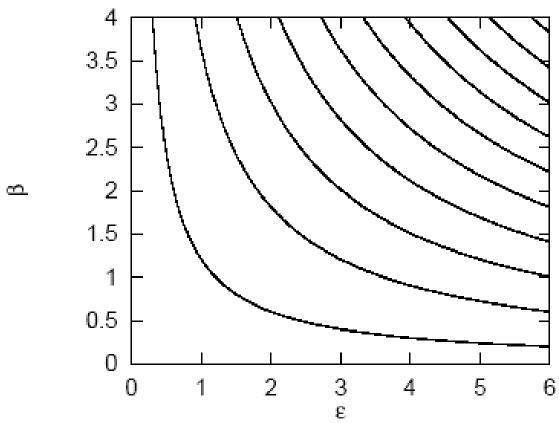

Figure 4. The phase diagram for the 1 dimensional irreps of the tetrahedral group T24.

$$\Lambda^{U(N)}_{\{\lambda\}} = \sum_{k=0}^{\infty}\frac{1}{k!}(\frac{\beta \in}{N})^k \det\{I_{\lambda_j - j + i - k}(\frac{\beta}{N}(1-\varepsilon))\} \quad . \tag{24}$$

The determinant picks up a set of {Λ} and k from the summation i.e. $\{\lambda_1 = \lambda_2 = ... = \lambda_N = k\} or \{l_1 = k + N - 1, ..., l_N = k\}$,

$$\Lambda^{U(N)}_{\{l\}} = \frac{(\frac{\beta}{N})^k}{k!} \quad . \tag{25}$$

Again a swap over between the largest Λ's happens at the point $\frac{\beta}{N} = k$,

$$\Lambda^{U(N)}_{\{l\}} = \frac{k^k}{k!} = \frac{k^{k-1}}{(k-1)!} = \Lambda^{U(N)}_{\{l'\}} > \Lambda^{U(N)}_{\{l''\}}, ... \tag{26}$$

Gauge theories with a local Z_n symmetry are of interest in the problem of quark confinement. The reason is that the center of the group SU(N) which is Z_n may be of particular importance in determining whether an SU(N) gauge theory is confining or not.

4. Numerical study

In this section we choose some non-Abelian and discrete sub-groups of SU(3) and by a numerical method plot their phase coexistence curves for an irreducible representation with aspecified dimension[8]. The numerical calculation is based on the eq. (4). At first the value of $\Lambda(B,\in)$ is calculated for different values of β and ε. The critical lines are identified as the points in which two largest Λs are equal. These points make critical curves on the β- ε phase diagram. The irreducible representations of these groups have different dimensionality and so we expect different behaviour for the phase diagrams. Actually the result of calculation for each group depends on the dimensionality of irreps that we use for the generalized action. The Z_3 group is the center of these groups and the phase diagrams of one and two dimensional irreps are similar to diagrams of the Z_3 group. Other irreps with higher dimensionality show slightly different pictures for the phase diagrams. We compared the phase diagrams of Z_3 , Z_4 , double

Table 1. Character table for the group T24 ($\omega=\exp(2\pi i/3)$). Each row corresponds to an irrep and each column to a class of the group elements.

1	1	1	1	1	1	1
1	1	1	ω	ω	ω^2	ω^2
1	1	1	ω^2	ω^2	ω	ω
2	-2	0	1	-1	1	-1
2	-2	0	ω	$-\omega$	ω^2	$-\omega^2$
2	-2	0	ω^2	$-\omega^2$	ω	$-\omega$
3	3	-1	0	0	0	0

Table 2. Character table for the group $\sum(216)$. This grop has got 216 elements($\omega=\exp(2\pi i/3)$). Each row corresponds to an irrep and each column to a class of the group elements.

1	1	1	1	1	1	1	1	1	1
1	ω	ω^2	1	ω	ω^2	1	1	ω	ω^2
1	ω^2	ω	1	ω^2	ω	1	1	ω^2	ω
2	-1	-1	0	1	1	-2	2	-1	-1
2	$-\omega$	$-\omega^2$	0	ω	ω^2	-2	2	$-\omega$	$-\omega^2$
2	$-\omega^2$	$-\omega$	0	ω^2	ω	-2	2	$-\omega^2$	$-\omega$
3	0	0	-1	0	0	3	3	0	0
8	2	2	0	0	0	0	-1	-1	-1
8	2ω	$2\omega^2$	0	0	0	0	-1	$-\omega$	$-\omega^2$
8	$2\omega^2$	2ω	0	0	0	0	-1	$-\omega^2$	$-\omega$

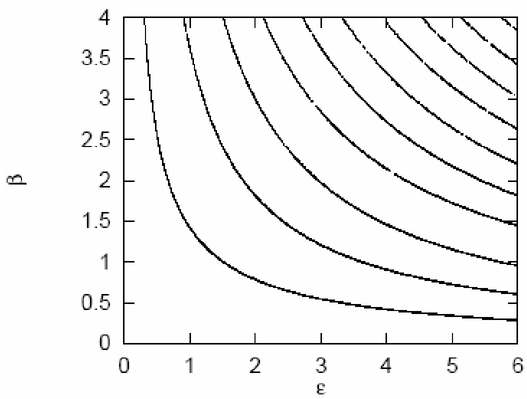

Figure 5. The phase diagram for the two dimensional irreps of the Σ(216).

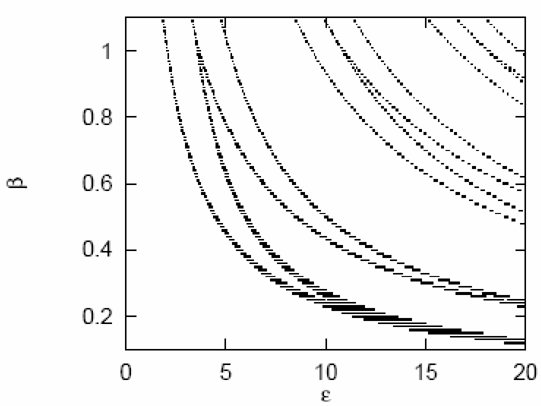

Figure 6. The phase diagram for the 8 dimensional irreps of Σ(216) The vertical line represent the real part of the gauge coupling constant and the horizontal line represent the imaginary part of the coupling constant.

tetrahedral T, Σ(36), Σ(168) and Σ(216); the phase diagram of Σ(216) is the interesting one and has got a tricritical point. The phase diagrams of Z_3 and T12 and Σ(216) are plotted in the following diagrams. Actually different behaviour of phase diagram corresponds to the dimension of the irreducible representation and for the 8 dimensional irreducible representation there is a tricritical point (figuer 3).

The phase diagram for a one dimensional irrep is plotted in the following picture. The phase diagram of the two dimensional irreps of Σ (216) (the fifth row in the character table of Σ (216)) is plotted in the following diagram. The phase diagram for the 8 dimensional irreps of the Σ(216) has a different behaviour compare to the other irreps with lower dimensions.

The periodic behaviour of the diagram comes from the imaginary part of the action. For small values of imaginary part the behaviour of the theory with the 8

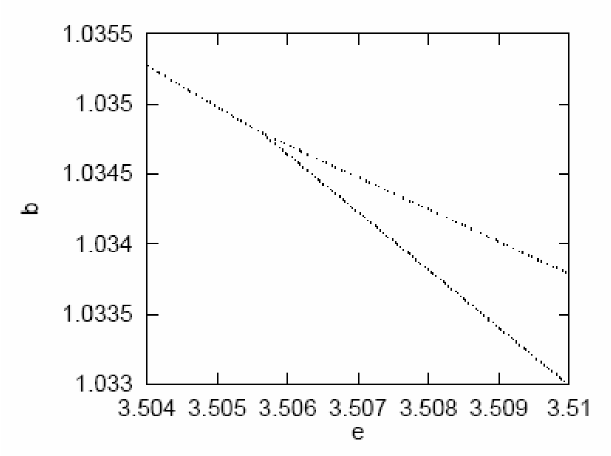

Figure 7. The three critical point for the Σ(216).

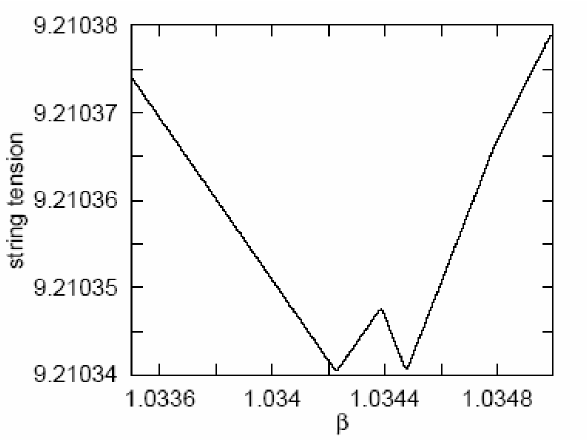

Figure 8. The string tension for a fixed value of ε=3.507

dimensional irreducible representation is similar to the one dimensional irreps or the center of Σ(216) which is Z_3.

There are three different phases around the three critical points. The critical lines are the places in the β- ε diagram which the largest Λ are equal this means that the string tension on these lines is zero (figure 8) The string tension does not vanish in a pure phase and the model is confining. The calculation of string tension is simple and is the same as finding the critical lines. After calculating the Λ's and sorting them numerically we have used (16) to get the value of string tension. The string tension for a fixed value of $\varepsilon = 3.507$ is plotted in the following picture $\beta \in [1.033, 1.0355]$.

In this section we have calculated the critical behaviour for different irreps of some discrete subgroups of SU(3) with a certain dimension. It should be noted that for understanding the full critical behaviour of these sub-groups one has to sum up over all dimensions, however we were interested only on behaviour of separate dimensions of irreps. The lesson one can learn from these is that the difference between the critical behaviour of the SU(3) and its center is due to contributions from the higher dimensional ($d \geq 2$) irreps.

5. Conclusion

The two dimensional pure gauge theory does not contain transverse propagating modes and is a trivial theory without phase transition. Generalization of the trivial action to a complex one leads to a theory with a rich phase structure. Study of this toy model is motivated by the results from the four dimensional effective pure gauge theory. In the real four dimensional theory after integrating out the fermionic degrees of freedom the effective action contains an imaginary part [6]. In general the same structure is expected for the SU(N) group. We could not solve the problem at large N limit. Actually there is a third order phase transition for large N limitof two dimensional gauge theory [9]. It is interesting to have a complete knowledge about the generalized action with SU(N) gauge group, and its large N limit. A good numerical algorithm for studying large N gauge theories can be found in [10].

Acknowledgements
It is pleasure to thank J.F. Wheater for helpful discussions.

References

1. G 't Hooft, *Nucl. Phys*. B**72** (1974) 461.
2. D Gross and W Taylor, *Nucl. Phys*. B**400** (1993) 181.
3. L N Lipatov, *Nucl. Phys.* B**365** (1991) 614.
4. J M Drouffe and J B Zuber, *Phys. Rept*. **102** (1983) 1.
5. T G Kovacs and J F Wheater, *Mod. Phys. Lett*. A **6**, **30** (1991) 2827.
6. L Alvarez-Gaume and E Witten, *Nucl. Phys*. B**234**, (1983) 269.
7. B Ye.Rusakov, *Mod. Phys. Lett*. A**5** (1990) 693.
8. W H Fairbairn et al., *J. of Math. Phys*. **5**, **8** (1964) 1038.
9. D Gross and E Witten, *Phys. Rev* D**21** (1980) 446.
10. T A Dickens et al., *Nucl. Phys*. B**309** (1988) 1-119.