

Spherical null shells within the distributional formalism

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Abstract

A null thin shell immersed in a generic spherically symmetric space-time is studied within the distributional formalism. It has been shown that the distributional formalism leads to the same result as the conventional Barrabes-Israel formalism.

Keywords: general relativity, null thin shell, distributional method

1. Introduction

Many practical problems of general relativity and cosmology involve idealized models constructed by gluing two regions with different metrics across a hypersurface or thin shell having a δ -function singularity in its Riemann tensor due to the discontinuity in the metric's transverse derivative across the shell. The description of timelike (or spacelike) thin shells is well known within general relativity since the outstanding work of Israel [1]. Later, an extension of the Israel formalism to the null or lightlike case was presented by Barrabes and Israel [2]. Very recently, Poisson has introduced a user-friendly reformulation of the Barrabes-Israel original work together with an illustration of the formalism [3].

In general, one can distinguish two equivalent approaches to describe thin shells or singular hypersurfaces as the boundary of two glued manifolds. The well-known Israel formalism relates the jump in the transverse curvature to properties of the singular hypersurface. This formalism being purely intrinsic, allows an independent and arbitrary choice of coordinates at both sides of the shell.

There is another formalism based on the distributional theory requiring a preconstruction of a common set of coordinates covering both sides of the shell making the four-metric continuous across the shell [4]. In this approach, for the non-lightlike case, it has been shown that the singularity in the Ricci part of the Riemann tensor is directly associated with the stress

energy tensor supported by the shell. The light-like case has recently been developed by Nozari and Mansouri [5], and applied to the spherically symmetric shell. They, however, have missed the construction of a suitable set of coordinates to make the four-metric continuous across the spherically symmetric layer, as it is required by the distributional formalism. Our task is to remedy this deficiency and incorporate an admissible continuous coordinate system across the null layer leading to the correct junction equations obtained in Ref. [2].

The plan of the paper is as follows. In section 2 we review shortly the distributional formalism for null shells and give the necessary formulae as a ready recipe to use. In section 3 we consider spherical null-like shells as a simple example to examine the efficiency of the distributional method.

Conventions: we use the metric signature $(- + + +)$, and define the Ricci tensor as $R_{\mu\nu} = \Gamma_{\mu\rho,\nu}^{\rho} - \dots$. The Greek indices run from 0 to 3 and Latin indices a, b from 1 to 3 but A and B take only values 2 and 3. The square brackets, [F], are used to indicate the jump of any quantity F across the layer. Terms proportional to δ -function are denoted by \check{F} .

2. Null-shell distributional formalism

Consider a space-time manifold \mathcal{M} consisting of overlapping domains \mathcal{M}_+ and \mathcal{M}_- with metrics $g_{\alpha\beta}^+(x_+^{\mu})$ and $g_{\alpha\beta}^-(x_-^{\mu})$ in terms of independent

disconnected charts x_+^μ and x_-^μ , respectively. The common boundary of the domains is denoted by Σ and taken to be light-like. In other words, the manifolds \mathcal{M}_+ and \mathcal{M}_- are glued together along the null hypersurface Σ . Introducing a single chart x^μ called admissible coordinate system that covers the overlap and reaches into both domain, we write down the parametric equation of Σ as $\Phi(x^\mu) = 0$, where Φ is a smooth function [2]. The domains of \mathcal{M} in which Φ is positive or negative are contained in \mathcal{M}_+ or \mathcal{M}_- respectively. By applying the coordinate transformations $x_\pm^\mu = x_\pm^\mu(x^\nu)$ on the corresponding domains, a pair of metrics $g_{\alpha\beta}^+(x^\mu)$ and $g_{\alpha\beta}^-(x^\mu)$ is formed over \mathcal{M}_+ and \mathcal{M}_- respectively, each suitably smooth (say C^3).

The main step in the distributional approach is the definition of a hybrid metric $g_{\alpha\beta}(x^\mu)$ over \mathcal{M} which glues the metrics $g_{\alpha\beta}^+(x^\mu)$ and $g_{\alpha\beta}^-(x^\mu)$ together continuously on Σ :

$$g_{\alpha\beta} = g_{\alpha\beta}^+ \theta(\Phi) + g_{\alpha\beta}^- (-\Phi), \quad (1)$$

where θ is the Heaviside step function and

$$[g_{\alpha\beta}(x^\mu)] = 0 \quad (2)$$

We expect on Σ the curvature and Ricci tensor to be proportional to δ function. It follows from eqs.(1) and (2) that the first derivative of $g_{\alpha\beta}$ is proportional to the step function. The δ distributional can only occur in the second derivative of the metric which enters linearly in the expression for curvature and Ricci tensor. So the only the relevant terms in the Ricci tensor are

$$\overset{\vee}{R}_{\mu\nu} = \overset{\vee}{\Gamma}_{\mu\rho,\nu}^\rho - \overset{\vee}{\Gamma}_{\mu\nu,\rho}^\rho. \quad (3)$$

Using the metric in the form (1), we finally arrive at the following expression for the components of the Ricci tensor proportional to δ distribution [4]

$$\overset{\vee}{R}_{\mu\nu} = \left(\frac{1}{2g} [g_{,\mu}] \partial_\nu \phi - [\Gamma_{\mu\nu}^\rho] \partial_\rho \phi \right) \delta(\Phi), \quad (4)$$

where g is the determinant of the metric and the partial derivatives are done with respect to the admissible coordinates x^μ .

The intrinsic coordinates of Σ adapted to its null generators are taken to be $\xi^a = (\eta, \theta^A)$, with η being an arbitrary parameter (not necessarily affine on either side of Σ) on the null generators of the hypersurface and θ^A as labels of the generators. Now we introduce

tangent vectors $e_a^\mu = \frac{\partial x^\mu}{\partial \xi^a}$, naturally segregated into a

null normal vector $\eta_\mu = \alpha^{-1} \partial_\mu \Phi$ that is also tangent to the generators, and two spacelike vectors e_A^μ pointing in directions transverse to the generators [3]

$$n^\mu = \left(\frac{\partial x^\mu}{\partial \eta} \right)_{\theta^A} = e_\eta^\mu, \quad e_A^\mu = \left(\frac{\partial x^\mu}{\partial \xi^A} \right)_\eta. \quad (5)$$

By construction, these vectors satisfy $n^\mu \eta_\mu = 0 = n_\mu e_A^\mu$.

We now complete the partial basis e_a^μ by adding a transverse null vector N^μ with the following properties [3]

$$N^\mu N_\mu = 0, \quad N_\mu n^\mu = -1, \quad N_\mu e_A^\mu = 0. \quad (6)$$

Therefore

$$\alpha = -N^\mu \partial_\mu \Phi. \quad (7)$$

The intrinsic metric on Σ may then be written as

$$\gamma_{AB} = g_{\mu\nu} e_A^\mu e_B^\nu, \quad (8)$$

and must be that same on the both sides of Σ . The Following jumps on Σ turn out to be vanishing:

$[\gamma_{AB}] = [n^\mu] = [e_A^\mu] = [N^\mu] = [\alpha] = 0$ expressed in the admissible coordinates x^μ .

The energy-momentum tensor of the shell $\overset{\vee}{T}_{\mu\nu}$, considered as a distribution, is given by [2-4]

$$\overset{\vee}{T}_{\mu\nu} = |\alpha| S_{\mu\nu} \delta(\Phi), \quad (9)$$

where $S_{\mu\nu}$ is the surface tensor of energy-momentum of the shell expressed in the admissible coordinates x^μ :

$$-\varepsilon S^{\mu\nu} = \sigma n^\mu n^\nu + j^A (n^\mu e_A^\nu + e_A^\mu n^\nu) + p \gamma^{AB} e_A^\mu e_B^\nu, \quad (10)$$

with $\varepsilon = \frac{|\alpha|}{\alpha}$. The first term represents a flow of matter along the null generators of the hypersurface, and hence σ represents a mass density. The second term represents a flow of matter in the direction transverse to the generators. Therefore j represents a current density. Finally, the surface quantity p represents an isotropic pressure.

Now we may write Einstein's field equation for the lightlike hypersurface Σ as follows [5]

$$\overset{\vee}{G}_{\mu\nu} = -\kappa \overset{\vee}{T}_{\mu\nu}. \quad (11)$$

Now taking into account eq. (4) we define

$$Q_{\mu\nu} = \left(\frac{1}{2g} [g_{,\mu}] \delta_\nu^\rho - [\Gamma_{\mu\nu}^\rho] \right) n_\rho. \quad (12)$$

Using eq.(9) for the energy momentum tensor we may write down eq. (11) as [5]

$$Q_{\mu\nu} - \frac{1}{2} g_{\mu\nu} Q = -\varepsilon \kappa S_{\mu\nu}, \quad (13)$$

where $Q = Q_{\mu\nu} g^{\mu\nu}$, and $Q_{\mu\nu}$ is a tensor with support on Σ . This so-called Sen equation obtained in the admissible coordinate system, describes the dynamics of

null surface layer Σ within the distributional approach.

3. Spherical null shells

To see the efficacy of the method, we consider the situation in which the null shell is immersed in a general spherical symmetric spacetime expressed in terms of the Eddington retarded or advanced time u [2]:

$$ds^2 = -e^\psi du (fe^\psi du + 2\zeta dr) + r^2 d\Omega^2, \quad (14)$$

where ψ_\pm and f_\pm are two arbitrary functions of the coordinates u_\pm and r_\pm . The sign factor ζ is +1(-1) if r increases (decreases) along a ray $u=\text{constant}$, i.e, if the light cone $u=\text{constant}$ is expanding (contracting). It is convenient to introduce the mass function $m_\pm(u_\pm, r_\pm)$

defined as $f = 1 - \frac{2m}{r}$. Consider now a thin spherical

shell whose history Σ being a light cone $u=\text{constant}$ splits the spacetime into the past-and future – domains \mathcal{M}_- and \mathcal{M}_+ . Our aim is to glue two space times \mathcal{M}_- and \mathcal{M}_+ along the hypersurface Σ using our distributional approach.

First we look for the admissible coordinate system $x^\mu = (u, r, \theta, \phi)$ in which the parametric equation describing the null shell is written as

$$\Phi(x^\mu) = u - R(r) = 0. \quad (15)$$

Now, we apply the following transformations to make the four – metric continuous on the shell.

$$\begin{aligned} u_- &= u, & u_+ &= A(u, r), \\ r_- &= r, & r_+ &= B(u, r). \end{aligned} \quad (16)$$

Carrying out the transformations and requiring the continuity of the metric on Σ according to eq. (2), we obtain

$$\begin{cases} U \equiv f_+ e^{2\psi_+} A_{,u}^2 + 2\zeta e^{\psi_+} A_{,u} B_{,u} = f_- e^{2\psi_-}, \\ X \equiv f_+ e^{2\psi_+} A_{,r}^2 + 2\zeta e^{\psi_+} A_{,r} B_{,r} = 0, \\ W \equiv f_+ e^{2\psi_+} A_{,u} A_{,r} + \zeta e^{\psi_+} (A_{,u} B_{,r} + A_{,r} B_{,u}) = \zeta e^{\psi_-}, \\ B(u, r) = r, \end{cases} \quad (17)$$

where \equiv means that both sides of the equality are evaluated on Σ . Taking $\eta = \zeta r$ to be the parameter on the null generators, from eq. (5) the tangent vectors are given:

$$n^\mu \partial_\mu = \zeta R_{,r} \partial_u + \zeta \partial_r, \quad e_\theta^\mu \partial_\mu = \partial_\theta, \quad e_\phi^\mu \partial_\mu = \partial_\phi. \quad (18)$$

According to eq. (8) the shell's intrinsic two – metric is given by

$$ds_\Sigma^2 = r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (19)$$

The continuity of the induced metric on Σ dictates the following condition on $R(r)$

$$R_{,r} = -2\zeta \frac{e^{\psi_-}}{f_-} \Big|_\Sigma \quad (20)$$

Note that in the admissible coordinate x^μ constructed by the transformation (16) the null hypersurface Σ is given by $fe^\psi du + 2\zeta dr = 0$. The components of the transverse null vector computed from eq. (6) are

$$N_\mu dx^\mu = \frac{1}{2} f_- e^{\psi_-} du \Big|_\Sigma. \quad (21)$$

Hence, using eq. (7) we get $\alpha = e^{-\psi_-}$. From the last equation of eq. (17) we obtain

$$f_- B_{,r} - 2\zeta e^{-\psi_-} B_{,u} = f_-. \quad (22)$$

Now, using eq. (22) the set of eqs. (17) can be solved for the unknown functions $A_u, A_r, B_u,$ and B_r :

$$\begin{cases} A_{,u} = -e^{\psi_- - \psi_+}, \\ A_{,r} = \frac{\Sigma - 2\zeta}{f_-} e^{-\psi_+}, \\ B_{,u} = \frac{\Sigma}{2} \zeta (f_+ - f_-) e^{\psi_-}, \\ B_{,r} = \frac{\Sigma f_+}{f_-}. \end{cases} \quad (23)$$

The nonvanishing components of $Q_{\mu\nu}$ computed from eq. (12) are

$$Q_{uu} = \frac{\zeta [f]}{r} e^{2\psi_-} - \zeta [\partial_r \psi] f_- e^{2\psi_-} \Big|_\Sigma, \quad (24)$$

$$Q_{rr} = 4\zeta \frac{[f]}{rf_-^2} \Big|_\Sigma, \quad (25)$$

$$Q_{ur} = Q_{ru} = \frac{2[f]}{rf_-} e^{\psi_-} - [\partial_r \psi] e^{\psi_-} \Big|_\Sigma, \quad (26)$$

where the derivatives in $[\partial_r \psi]$ are taken with respect to the relevant radial coordinates r_\pm . Now, we can immediately calculate Q as

$$Q = 2\zeta [\partial_r \psi] \Big|_\Sigma. \quad (27)$$

The nonzero components of the surface energy tensor $S^{\mu\nu}$ are then calculated to be

$$S_{uu} = -\sigma e^{2\psi_-} \Big|_\Sigma, \quad (28)$$

$$S_{rr} = -4\sigma f_-^{-2} \Big|_\Sigma. \quad (29)$$

$$S_{ur} = -2\zeta \frac{\sigma}{f_-} e^{\psi_-} \Big|_\Sigma, \quad (30)$$

$$S_{\theta\theta} = -pr^2 \Big|_\Sigma, \quad S_{\phi\phi} = \sin^2 \theta S_{\theta\theta}. \quad (31)$$

Finally, we obtain the following junction equation from the non – angular components of eq. (13)

$$\sigma = (-\zeta) \frac{[m]}{4\pi r^2} \Big|_\Sigma. \quad (32)$$

From eq. (32) we see that for a positive mass density σ if $\zeta > 0$, then $[m] < 0$ meaning the shell extracts energy

from the center, whereas $[m] > 0$ for $\zeta < 0$, means that the shell brings energy to the center. The angular components of eq. (13) yield the another junction equation as

$$p = (-\zeta) \frac{1}{8\pi} [\partial_r \psi]_{\Sigma}, \quad (33)$$

which expresses the first law of thermodynamics on the null shell.

These junction equations are the same as those obtained by C. Barrabes and W. Israel (see eqs (51) in Ref. [2]).

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4. Conclusion

Direct application of the distributional formalism to describe thin layers in general relativity requires a preconstruction of spacetime coordinates that match continuously on the shell to make the four-metric continuous. We have explicitly shown that in the case of a null shell embedded in a generic spherically symmetric geometry, it is possible to construct such an admissible coordinate system covering both sides of shell, leading to the same result as the common Barrabes–Israel formalism. This construction allows us to apply the distributional formulation to treat null surface layers in general relativity.