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On synchronization of clocks in general space-times

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Abstract

Einstein and transport synchronizations of infinitesimally spaced and distant clocks are considered in a general Riemannian space-time. It is shown that infinitesimally spaced clocks can always be synchronized. In general one can not find observers for whom distant clock are Einstein synchronized; but transport synchronized observers do always exit. Whenever both procedures are possible, they are equivalent.

Keywords: relativity, synchronization, riemannian, space-time

1. Introduction

The possibility of very accurate measurements of time has been of utmost importance in experimental relativity [1]. Many advances have been made in this direction during the last two decades, and measurements of 10^{-10} seconds accuracy have been achieved. As a consequence of these developments, not only the standard tests of the relativity theories are pushed further in precision but the measurement of certain other relativistic effects on the surface of the earth and in space are made possible [2,3]. These effects are due to the rotation and the gravitational field of the earth as well as the motion of clock transported in a gravitational field. Clear definitions of synchronization and synchronization procedure are required for any meaningful discussion of these effects. Hence the synchronization procedures and their ramifications, which previously were only of theoretical interest, have now become an important experimental matter.

It is rather well known that distant identical clocks can be synchronized either by Einstein procedure (E-synchronization) or by slow transport method (T-synchronization). By identical clocks we mean clocks having the same rates when brought to the same place. E-synchronization is synchronization by electromagnetic signals. There is an excellent discussion of this method, for infinitesimally spaced clocks in Riemannian space, in Landau and Lifshits [4]. They, however, clearly state that E-synchronization of distant clocks in an stationary metric is not always possible. Synchronization by slow transport of a clock is seldom discussed in the standard texts of relativity, and certainly not in Landau and Lifshits. Eddington mentions it and shows the equivalence of two procedures of synchronization in special relativity, where only inertial observers and clocks stationary with respect to them are involved [5].

Mansouri and Sexl have shown that the equivalence of E- and T-synchronization of clocks in special relativity provides us with a means to test the fundamental symmetry of the underlying space time structure of special relativity (SR) [6]. One can devise rival theories leading to the same standard results as SR, while breaking Lorentz invariance. For these theories, E-synchronized clocks will read differently from T-synchronized ones. By parametrizing the family of rival theories, one can argue how accurately SR is corroborated by standard tests. Such a scheme is called a test theory of special relativity.

The first detailed discussion of these synchronization procedures in general relativity is due to Ashby and Allan in their pioneering work [3]. In order to compare remote clocks as well as to make necessary corrections in readings of the clocks transported across the Atlantic, they had to make a careful analysis of different relativistic effects of the gravitational field of the rotating earth. The inaccurate statements and erroneous predictions seen occasionally in the literature regarding the E- and T-synchronizations of clocks around the earth indicate that such discussions should

not be considered trivial at all [7].

A first step towards clarifying these points in a general framework is the discussion of synchronization on a rotating disc, as a model for the rotating earth [8]. It is shown that a one-parameter test theory of special relativity on the rotating frame can be formulated. In section 2.1 we discuss this special case briefly. To synchronize the clock, by whichever procedure, we have to correct the reading of the clocks at each point. It turns out that on the rotating disc the corrections needed are the same for both methods. One should note that the "synchronization" of distant clocks on the rotating disc is path-dependent for both procedures. Therefore the above-mentioned equivalence actually holds only for the same synchronization path. While we do not have path-independence and global synchronization on the rotating disc we can transform to a coordinate system (Lorentz frame) where all clocks are synchronized by both procedures, independent of the path of synchronization. This may not be so in a general Riemanian space-time. The aim of the paper is to elucidate the problem of synchronization in a general space-time. One should note that the entire discussion of the present paper can be formalized in mathematical terms using a congruence of observers. But we believe that nothing is gained by such a formal approach and one may lose the physical intuition leading to the point mentioned in the concluding section of this paper.

Rumpf [9] studied the synchronization of clocks in an arbitrary stationary space-time. He showed that the two procedures are equivalent provided "the clocks that are to be synchronized follow trajectories of a time-like Killing vector", and "the synchronization paths [of the light ray and the synchronizing clock] are the same." In sections 2.2, 2.3, and 2.4 we consider several examples to clarify the observer-dependence of synchronization process, and its implications. These examples are Schwartzschild, Kerr and FRW space-times. In section 3, E- and T-synchronization of *infinitesimally spaced* as well as *distant* clocks are discussed in a general space-time. The main results of this paper are (i) the transport synchronization of clocks is the more fundemental one of the two procedures, and (ii) in *any* Riemannian space-time, whenever the T- and E-synchronizations are possible, these two methods lead to equivalent results. Therefore any test of relativity based on a comparison of these methods is a test of Riemannian structure of space-time.

2. Specific examples

Before discussing specific examples, we think it appropriate to clarify certain elementary notions, which will be used afterwards. By choosing a coordinate system, we define a certain time and a certain sort of synchronization. The question here is whether the clocks showing this coordinate time are T-(or E-) synchronized. It turns out that one should know something about the workings of these clocks, or rather how are they related to natural clocks, to be able to discuss T-synchronization [10]. Consider a space-time which is asymptotically flat. Take a natural clock at infinity and one at point x. The clock at infinity reads the coordinate time which is the same as the natural time at infinity. Assume that this clock sends signals at unit time intervals. The "clock" at x is provided with a mechanism that manipulates its natural rate such that the signals are received at unit time intervals. Such a clock is a "coordinate clock". Its rate is given by dt. The rate of a fixed natural clock at x is given by ds, which differs from dt by the factor $1/\sqrt{g_{00}}$. Assuming that we have only natural clocks, we can divide the rate of natural clocks at any point and any moment by $\sqrt{g_{00}}$ and obtain the rate of the coordinate clocks without resorting to the clocks at infinity. Such a device reading coordinate time can be defined for any arbitrary space-time even if it is not asymptotically flat. Its reading is given by

$$t = t_0 + \int_{s_0}^{s} ds/\sqrt{g_{00}}\,, \tag{1}$$

where the setting t_0 (and s_0) is arbitrary. However if we have a network of coordinate clocks, we should synchronize the settings according to a certain procedure.

2.1. Rotating Disc

The metric on the rotating disc can be written as [4]

$$ds^2 = dt^2 - r^2 d\phi^2 - \frac{2\omega}{\sqrt{1-r^2\omega^2}} r^2 dt d\phi - dr^2\,, \tag{2}$$

where t is the proper time of the clock sitting on the rotating disc; ω is the angular velocity of the disc and ϕ the usual polar coordinate. First consider two clocks A and C [fig.1] lying on a radius $(d\phi = 0)$. A light signal sent from A to C (and back) has the same transit time in both directions. Therefore they are E-synchronised. A slowly transported clock $((dr/dt)^2 \langle\langle 1)$ along the same path reads always the time t. So clocks A and C are also transport synchronised.

Now consider clocks A and B on the circular path (figure 1). It is well known that the clocks are not synchronised and we have to make corrections. The corrections for both procedures (Einstein and transport) are the same and equal to [8]

$$\Delta t_1 = \frac{\pi\ r_1^2\omega}{\sqrt{1-r_1^2\omega^2}}\,. \tag{3}$$

If we know synchronised clocks A and B along path 2 (ACDB) the correction again is the same and equal to

$$\Delta t_2 = \frac{\pi\ r_2^2\omega}{\sqrt{1-r_2^2\omega^2}}\,, \tag{4}$$

which is different from Δt_1 i.e the corrections are path-dependent.

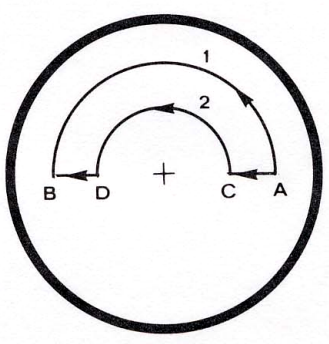

Figure 1. Clocks A and B on the circular path.

Metric (2) can be considered as a flat spacetime for a non-inertial observer. It is obvious that we can transform back to an inertial frame where all clocks are always Einstein- as well as transport-synchronized, independent of the path.

2.2. Schwartzchild space-time

The metric of the spherically symmetric spacetime in Schwarzschild coordinates is written as

$$ds^2 = (1-2M/r)dt^2 - \frac{1}{1-2M/r}dr^2 - r^2 d\Omega^2 = \quad (5)$$
$$(1-2M/r)dt^2 - dl^2$$

This metric is in Gaussian form, which means that the signal transit time between two infinitesimally spaced points is independent of direction if time is measured by coordinate clocks reading *t*. A natural clock fixed at a point reads the proper time interval

$$ds = \sqrt{1-2M/r}\,dt \quad (6)$$

Therefore when measuring the speed of light using natural clocks we have to correct the rate by $1/\sqrt{1-2M/r}$.

Two coordinate clocks, infinitesimally spaced from each other, are obviously E-synchronized. A slowly transported natural clock $\left((dl/dt)^2 \langle\langle 1\right)$ reads the same time interval as eq. (6). Therefore all infinitesimally spaced clocks are also T-synchronized. This shows the equivalence of the two procedures for two infinitesimally spaced clocks.

Now we consider distant clocks A and B. As can be seen from eq. (6) for any path the transit time for a signal to go from A to B is the same as from B to A. Clocks A and B are therefore E-synchronized and this synchronization is path independent. Are these clocks also transport synchronized? The answer is in positive, provided the slowly transported clock is a coordinate clock. Therefore E- and T-synchronization of distant clocks are equivalent in Schwartzschild coordinate.

This equivalence does not however hold if we choose another coordinate system. Consider for example the radially free falling observers. In this case the Schwartzschild metric can be written as [11]

$$ds^2 = d\tau^2 - [\frac{3}{2}(\frac{R-\tau}{2M})]^{3/2} dR^2 - \quad (7)$$
$$[\frac{3}{2}(R-\tau)]^{4/3}(2M)^{2/3} d\Omega^2 \ ,$$

which is obtained using the transformation
$\tau = t - f(r)$

$$R = t - f(r) + (2/3)(2M)^{-1/2} r^{3/2} \quad (8)$$

where

$$\frac{\partial f}{\partial r} = \sqrt{\frac{2M/r}{1-2M/r}} \ .$$

Note that

$$r = (2M)^{1/3}[\frac{3}{2}(R-\tau)]^{2/3} \ . \quad (9)$$

Consider two special cases:

i) Distant clocks A and B are fixed at two points of a radial null-geodesics $d\phi = d\theta = 0$. These are freely falling clocks. Speed of light $dr/d\tau$ along this null-geodesic depends on time τ , which means the clocks are not E-synchronized. On the other hand all natural clocks are coordinate clocks and since there is no cross term involving $d\tau$ in the metric, the T-synchronization is guaranteed for all distant clocks. Here is a case where clocks are T-synchronized but not E-synchronized. We shall show later that in a general space-time written in Gaussian normal coordinates the T-synchronization of distant clocks is always guaranteed, while no such statement can be made for E-synchronization.

ii) Two distant clocks A and B fixed at $R_A = R_B = R_0$ with arbitrary θ and ϕ . The light velocity along any path between A and B depends on time τ . Therefore A and B are not E-synchronized.

From cases (i) and (ii) it results that no two clocks located at arbitrary points of (τ, R, θ, ϕ) frame are E-synchronized. Usually when we talk of synchronization of clocks, we assume that they are fixed in the corresponding coordinates. Let us assume that the two distant clocks A and B are moving in (τ, R) -frame in such a way that both have fixed position in (r,t) -frame. A sends a signal at $t_1(\tau_1)$ which is received by B at $t_2(\tau_2)$ and is immediately sent back to A, arriving there at $t_3(\tau_3)$ (figure 2). The clocks are E-synchronized in (t,r) coordinate, so we have

$$t_2 - t_1 = t_3 - t_2 \ . \quad (10)$$

Now the question is whether the clocks are synchronized in (τ, R) -frame i.e.

$\tau_2 - \tau_1 = \tau_3 - \tau_2$. From eq. (8) we have

$$(t_2 - t_1) - [f(r_B) - f(r_A)] = (t_3 - t_1) - [f(r_A) - f(r_B)] \ . \quad (11)$$

It then follows that

$$f(r_A) = f(r_B) \ , \quad (12)$$

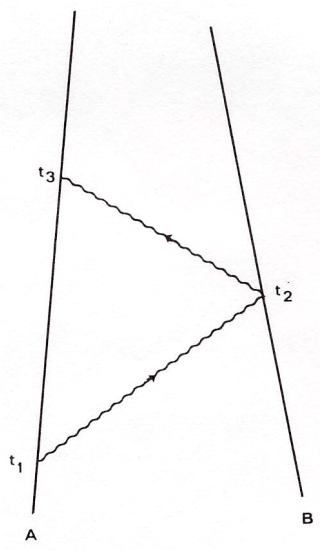

Figure 2. Sending and receiving signals between clocks A and B.

or $r_A = r_B$. That is clocks A and B are "E-synchronized" in (τ, R) if and only if they are fixed at the same $r_A = r_B = r_0$ or $R - \tau = r_0$ i.e. clocks A and B are moving in (τ, R).

2.3. Kerr space-time

The metric of Kerr space-time in Boyer-Lindquist coordinates has the form[12]:

$$ds^2 = \frac{1-2M/r+(a^2/r^2)\cos^2\theta}{1+(a^2/r^2)\cos^2\theta}dt^2 + \frac{2M(a/r)\sin^2\theta}{1+(a^2/r^2)\cos^2\theta}dtd\phi - \frac{1+(a^2/r^2)\sin^2\theta}{1-2M+a^2/r^2}dr^2 - r^2[1+(a^2/r^2)\cos^2\theta]d\theta^2 - \frac{r^2\sin^2\theta}{1+(a^2/r^2)\cos^2\theta}[(1+a^2/r^2)^2 - a^2/r^2(1-2M/r+a^2/r^2)]d\phi^2 \ . \quad (13)$$

The metric coefficients are independent of time; but there is a cross term $dtd\phi$. In general distant clocks are not T- or E-synchronized. However, clocks lying on $\phi =$ surfaces are both E- and T-synchronized. The coordinate clocks lying on the equatorial plane $(\theta = \pi/2)$ are not synchronized. We can make them synchronized by correcting the time by an amount

$$\delta t = -\frac{4Ma}{r(1-2M/r)}d\phi \ . \quad (14)$$

This correction makes the clocks T-synchronized too. But it should be noted that this "synchronization" is path-dependent.

2.4. Friedmann-Robertson-Walker space-time

FRW metric is written as

$$ds^2 = dt^2 - R(t)[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)] \ . \quad (15)$$

Obviously, the distant clocks are T-synchronized. Slowly moving clocks read the coordinate time, which is usually called cosmic time. However, distant clocks are not E-synchronized except when $K = 0$, and that in the following sense: Consider two distant clocks A and B. A signal is sent from A to B and back to A. The time taken for the signal to go from A to B is different from the time of return travel even when $K = 0$. However, if two signals are sent at the same coordinate time *t*, one from A to B and the other from B to A, they would be received simultaneously by A and B when $K = 0$. It is in this sense that we mean the clocks are E-synchronized.

FRW space-time may also be written in the conformally flat form [12]

$$ds^2 = a^2(\tau)[d\tau^2 - d\sigma^2], \quad (16)$$

where $dt = R(t)d\tau = a(\tau)d\tau$ and $d\sigma^2$ is the space part of the metric. Here the distant clocks are E-synchronized but they are only T-synchronized if they are corrected such that they read the coordinate time.

3. General space-time

We write the metric of a general Riemannian space-time as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{00}dt^2 + 2g_{0i}dtdx^i + g_{ij}dx^i dx^j \quad (17)$$

In this metric, stationary clocks are not in general E- or T-synchronized. But for infinitesimally spaced clocks we introduce a correction

$$\delta t_E = -\frac{g_{0i}}{g_{00}}dx^i \ , \quad (18)$$

so that they are E-synchronized [4]. Slowly transported coordinate clocks read $ds/\sqrt{g_{00}}$, which is equal to

$$\frac{ds}{\sqrt{g_{00}}} = dt[1-(g_{0i}/g_{00})\frac{dx^i}{dt}] \ . \quad (19)$$

Therefore the correction needed to make these clocks synchronized is the same as δt_E above: $\delta t_E = \delta t_T$. So, infinitesimally spaced clocks can be made both E- and T-synchronized.

Distant clocks are obviously neither T- nor E-synchronized in general. The necessary condition for realization of T- or E-synchronization of clocks is to have a Gaussian coordinate system where no space time cross term exists in the metric,

$$ds^2 = \bar{g}_{00}dt^2 - \bar{g}_{ij}dx^i dx^j \ . \quad (20)$$

Here the coordinate clocks are T-synchronized. If we want to have the natural clocks T-synchronized, we should transform to Gaussian normal coordinates,

$$ds^2 = d\tau^2 - g_{ij}(x,t)dx^i dx^j \quad (21)$$

which correspond to observers defined by a hypersurface orthogonal time-like vector field with unit norm.

In general, due to the time and space dependence of g_{ij} and $\bar{g}_{ij}$ in (20) and (21), the transit time of a signal going from A to B is different from that of a signal going from B to A. Therefore clocks can not be E-synchronized unless the space-time is static. However there may exist specific paths (e.g. $\phi = const$ paths in Kerr space-time) along which E-synchronization is possible.

4. Conclusions

1. The infinitesimally spaced clocks can always be made E- and T-synchronized by the same amount of correction, i.e. the two types of synchronization are equivalent.
2. Distant clocks, in general space-time, are not E- or T-synchronized for any obsever. They are, however, T-synchronized for observes defined by the Gaussian normal coordinates.
3. In general space-time there may exist paths or sheets on which the clocks are E-synchronized.
4. In any static space-time, for observers defined by Gaussian normal coordinates, distant clocks are E- and T-synchronized.
5. Whenever E- and T-synchronization of distant clocks are possible for an observer, they are equivalent.

Regarding the equivalence of E- and T-synchronization of infinitesimally spaced clocks (conclusion 1 above) the following discussion may be of interest. This equivalence is essentially a consequence of the Riemannian structure of the space-time. One may look the other way round, and use this equivalence to test the space-time structure experimentally. A proper approach to implement it is to assume a more general space-time, say a Finslerian space, and investigate the consequences related to different synchronization methods. One may be able to formulate a test theory of Riemannian structure of space-time more or less in the spirit of the test theory of SR, which is recently cast into the geometrical structure of a Finslerian space[13,14].

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