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A three-body force model for the harmonic and anharmonic oscillator

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Abstract

We present a mathematical method to describe motion of a system based on 3 identical body forces. The 3-body forces are more easily introduced and treated within the hyperspherical harmonics. We have obtained an exact solution of the radial Schrödinger equation for a 3- body system in three dimensions. The interact potential V is assumed to depend on the hyperradius x only where x is a function of the Jacobi relative coordinates ρ and λ which are functions of the three identical particles, relative positions $\vec{r}_{12}, \vec{r}_{23}$ and $\vec{r}_{31}$. This method has been extensively used in nuclear and molecular physics. This work is interesting to those who are studying hadronic and bosonics physics and problem consisting three - body systems.

Keywords: hypercentral - three-body- Schrödinger- hypersphreical, Jacobi relative coordinates

1. Introduction

The three body-forces are more easily introduced and treated within the hyperspherical harmonics formalism [1,2 and 3]. Introducing the center-of-mass coordinate R and the Jacobi relative coordinates ρ and λ

$$R = \frac{(r_1 + r_2 + r_3)}{3} \quad \rho = \frac{(r_1 - r_2)}{\sqrt{2}} \quad \lambda = \frac{(r_1 + r_2 - 2r_3)}{\sqrt{6}} , \tag{1}$$

and the conjugate momenta P_R , P_ρ and P_λ the kinetic energy becomes

$$E_c = 3m + \frac{P_R^2}{2m} + \frac{(P_\rho^2 + P_\lambda^2)}{2m} .$$

The three –quark space wavefunction, in agreement with translational invariance, is

$$\psi(\vec{r}_1, \vec{r}_2, \vec{r}_3) = (2\pi)^{-3/2} e^{i\vec{P}_R \cdot \vec{R}} \psi(\rho, \lambda) ,$$

where r_1, r_2 and r_3 are the positions of three identical particles.

The Jacobi coordinates have many applications, one of them being bosonic quantization. The method of bosonic quantization consists of two vector boson operators (one for each relative coordinate) which are related to the coordinates, ρ and λ and their conjugate momenta, P_ρ and P_λ, by

$$\begin{aligned}
b_{\rho,m} &= \frac{1}{\sqrt{2}}(\rho_m - iP_{\rho,m}), \\
b^{-}_{\rho,m} &= \frac{1}{\sqrt{2}}(\rho_m + iP_{\rho,m}), \\
b_{\lambda,m} &= \frac{1}{\sqrt{2}}(\lambda_m - iP_{\lambda,m}), \\
b^{-}_{\lambda,m} &= \frac{1}{\sqrt{2}}(\lambda_m + iP_{\lambda,m}),
\end{aligned} \tag{2}$$

with $m = -1, 0, 1$ these operators satisfy usual boson commutation relations and operators of different types of commutes. The nonrelativistic harmonic oscillator quark model [4] is a model of this type, although it is written for the Hamiltonian.

$$\begin{aligned}
H &= \frac{P_\rho^2}{2m} + \frac{P_\lambda^2}{2m} + \frac{3}{2} k \rho^2 + \frac{3}{2} k \lambda^2 + \text{perturbations} \\
&= \varepsilon(-b_\rho \cdot \tilde{b}_\rho - b_\lambda \cdot \tilde{b}_\lambda + 3) + \text{perturbations} \\
&= \varepsilon(n_p + n_\lambda - 3) + \text{ perturbations},
\end{aligned} \tag{3}$$

with $\varepsilon=\sqrt{\frac{3k}{m}}$.The perturbations involve both anharmonic terms and terms that couple different shells. To solve the equation analytically, let's define the hypercentral coordinates.

The two Jacobi coordinates ρ and λ are relevant degrees of freedom (in addition the center-of-mass coordinate is not relevant). The hypercentral coordinates are defined in terms of the absolute values ρ and λ

$$x=\sqrt{\rho^2+\lambda^2}=\sqrt{\frac{1}{3}(r_{12}^2+r_{23}^2+r_{31}^2)} \qquad t=arctg\,(\frac{\rho}{\lambda}), \tag{4}$$

where x is the hyperradius and is a function of r_1, r_2 and r_3 the three identical particle relative positions and t is the hyperangle, together with the angles $\Omega_\rho, \Omega_\lambda$. After having separated the c.m. motion $\vec{R}$ the Laplace operator for the three particles system becomes ($\hbar=c=1$)

$$(\nabla_\rho^2+\nabla_\lambda^2)=(\frac{\partial^2}{\partial x^2}+\frac{5}{x}\frac{\partial}{\partial x}+\frac{L^2(\Omega)}{x^2}), \tag{5}$$

where $\frac{L^2(\Omega)}{x^2}$ is a generalization of the centrifugal barrier for the case of six dimensions and it involves the angular coordinates $\Omega_\rho, \Omega_\lambda$ and the hyperangle t.

The eigenvalues of $L^2(\Omega)$ are given $L^2(\Omega)=-\gamma(\gamma+4)$ where γ is the grand angular quantum number, using standard notation, the principal quantum numbers of the ρ-oscillator is $N_\rho=(2n_\rho+l_\rho)$, and similarly for the λ-oscillator. The energy of a state specified by the quantum number N given by $2n+l_\rho+l_\lambda$,

$$E_n=(N+\frac{3}{2})\omega\ , N=N_\rho+N_\lambda$$
$$=(2n_\rho+l_\rho)+(2n_\lambda+l_\lambda)=2n+l_\rho+l_\lambda\ ,$$

(6) where n is a positive integer and l_ρ and l_λ are the angular momentums corresponding to ρ and λ.

For a given value of $n, i.e.$ the model space in which calculation are done, one has

$$\begin{aligned}
&n_\rho=0,1,......n\\
&n_{\rho_\lambda}=0,1,......n-n_\rho\\
&l_\rho=n_\rho, n_\rho-2,...,1\ or\ 0,\\
&l_\lambda=n_\lambda, n_\lambda-2,...,1\ or\ 0\\
&l=\left|l_\rho-l_\lambda\right|,\left|l_\rho-l_\lambda\right|+1,...l_\rho+l_\lambda,\\
&m_l=-l, l+1,..,l
\end{aligned} \tag{7}$$

The parity of the state is $\pi=(-)^{l_\rho+l_\lambda}$. The basis states are then uniquely labeled by

$$\left|n,(n_\rho,l_\rho),(n_\lambda l_\lambda);l,m_l\right\rangle \tag{8}$$

The same basis of two coupled harmonic oscillators is employed in the nonrelativistic and relativistic quark models. Early quark model calculations [4] used $n_\rho+n_\lambda\leq 2$, while more recent calculations [4] have used $n_\rho+n_\lambda\leq 6$. The eigenfunctions of the grand-angular operator $L^2(\Omega)$ are denoted by $Y_{[\gamma]}(\Omega_\rho,\ \Omega_\lambda,\ t)$ and are known as the products of spherical harmonics with angular momentums l_ρ and l_λ and of Jacobi polynomials in the hyperangle t [3]. They are called hyperspherical harmonics and form a complete orthogonal basis in the space of function of $\Omega_\rho, \Omega_\lambda, t$.

2. An exact solution of the three-body Schrödinger wave equation for a sextic potential

In general the space part of the three particles wave function is expanded in the hyperspherical harmonics basis and the Schrödinger equation leads to a set of coupled differential equations [4,5]. That is, the assumption that for each body consists of three identical like baryon state, only one hyperspherical harmonic is sufficient. In this respect, it is interesting to observe that the matrix elements of the currently used two-body potentials in the three-body agree almost perfectly with this hypercentral behavior [5,6].

On the other hand, if the potential $V(x)$ is assumed to depend on the hyperradius x only, the space wave function is factorized similarly to the central potential case. The potential $V(x)$ is called hypercentral, in the sense that it is invariant for any rotation in the 6-dimensional space spanned by the coordinates (O (6) symmetry). The dependence on x means in general that the potential has a three-body character, since the dependence on the single pair coordinates cannot be disentangled from the third one. The hyperradial wave functions $\psi_{\nu\gamma}(x)$ is a solution of the reduced Schrodinger equation:

$$-\frac{1}{2m}\left[\frac{d^2\psi_{\nu\gamma}(x)}{dx^2}+\frac{5}{x}\frac{d\psi_{\nu\gamma}(x)}{dx}-\frac{\gamma(\gamma+4)}{x^2}\psi_{\nu\gamma}(x)\right] \tag{9}$$
$$+V(x)\psi_{\nu\gamma}(x)=E\psi_{\nu\gamma}(x)\ ,$$

where m is the particle mass. For a fixed γ there are different solutions, which can be labeled by ν; where $\nu+1$ is the number of nodes of the wave function. The h.o potential has a two-body character, but it can be treated by means of the hypercentral eq. (9) since

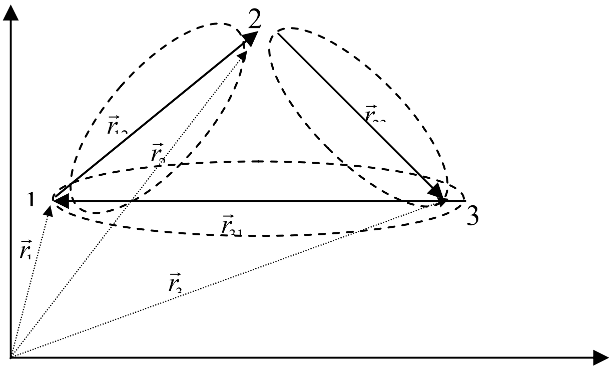

Figure 1. Diagram of three body interaction.

$$V_{h.o}(\rho,\lambda)=\frac{1}{2}\sum_{i<j}k(r_i-r_j)^2= \qquad (10)$$

$$\frac{1}{2}k(r_{12}^2+r_{23}^2+r_{31}^2)=\frac{3}{2}kx^2$$

where $x^2=(\rho^2+\lambda^2)=\frac{1}{3}(r_{12}^2+r_{23}^2+r_{31}^2)$ and so it is exactly hypercentral.(Figure 1) eq. (9) has an analytical solution [7] also for the sextic potential $V_{hyc}(x)=bx^4+cx^6$.

Using an ansatz for the eigenfunctions, we obtain an exact analytical solution of the Schrodinger wave equation for the doubly anharmonic (sextic) three body potential, $V(x)=ax^2+bx^4+cx^6$, where a,b and c are positive and satisfy the constraint $\frac{b^2}{4c}=(\frac{c}{2m})^{1/2}(2\gamma+8)+a$, with as a ground orbital quantum number. The problem of the anharmonic oscillator with quartic type anharmonicity in the two potential has been very widely studied in different contexts and also at the level of both classical and quantum mechanics. Only recently, it seems to be newly discovered [8,9] phenomena (such as structural phase transitions [8], polar on formation in solids [10], the concept of false vacua in field theory [11]) whose theoretical understanding might require the introduction of higher order anharmonicity in the potential, particularly that of sextic type. Unfortunately not much work has been carried out on the doubly anharmonic (sextic) potential except for some studies [12] at the classical level for two body problem. In this note, we present an exact analytic solution of the radial Schrodinger wave equation [13] for three body problem which its results can be used for two body problem as well. For a sextic three-body potential in three dimension of the type

$$V(x)=ax^2+bx^4+cx^6\ , \qquad (11)$$

where a, b and c are positive and satisfy a constraining condition (cf. eq(21) below).

First of all the transformation $\psi_{\nu\gamma}(x)=x^{-\frac{5}{2}}\varphi_{\nu\gamma}$ reduces (4) to the form

$$\varphi''_{\nu\gamma}(x)+(\varepsilon-a_1x^2-b_1x^4-c_1x^6 -\frac{(2\gamma+3)(2\gamma+5)}{4x^2})\varphi_{\nu\gamma}(x)=0 \qquad (12)$$

where

$$\varepsilon=2mE\,,\ a_1=2ma,\ b_1=2mb,\ c_1=2mc\ , \qquad (13)$$

Now, for the eigenfunctions $\varphi_{\nu\gamma}(x)$ we make an ansatz for the wavefunction [14,15,16 and 17]

$$\varphi_{\nu,\gamma}(x)=f(x)\exp\left[g(x)\right]\ , \qquad (14)$$

where

$$f(x)=\prod_{i=1}^{N}(x-\alpha_i^n) \qquad N=1,2,\ldots$$

$$f(x)=1 \qquad N=0$$

$$g(x)=-\frac{1}{2}\beta x^2-\frac{1}{4}\alpha x^4+\delta\ln x\ , \qquad (15)$$

which implies that

$$\varphi''(x)=(g''(x)+g'(x)^2 +\frac{f''(x)+2g'(x)f'(x)}{f(x)})\varphi(x)\ ,$$

(16) or for the ground state $f(x)=1$

$$\varphi''(x)+\left[\begin{matrix}+(2\delta+1)\beta+(3\alpha+2\alpha\delta-\beta^2)x^2\\ -2\alpha\beta x^4-\alpha^2x^6-\delta(\delta-1)/x^2\end{matrix}\right]\varphi(x)=0\ . \qquad (17)$$

On comparing eqs. (17) and (12) we obtain

$$\varepsilon=+(2\delta+1)\beta,\beta^2-3\alpha-2\alpha\delta=a_1-2\alpha\beta=b_1,$$
$$\alpha^2=c_1,\ \delta(\delta-1)=(\gamma+\frac{3}{2})(\gamma+\frac{5}{2})\ . \qquad (18)$$

These equations yield

$$\alpha=\sqrt{c_1}\ ,\beta=\frac{-b_1}{2\sqrt{c_1}},\ \delta=-\gamma-\frac{3}{2},\delta=\gamma+\frac{5}{2}\ . \qquad (19)$$

Here we shall use only the second value of $\delta=\gamma+\frac{5}{2}$ as it provides a well-behaved solution at the origin. The ground state eigenvalues ε can be obtained from (18) as

$$\varepsilon_\gamma=(\gamma+3)\frac{b_1}{\sqrt{c_1}}=2(\gamma+3)\left[c_1^{\frac{1}{2}}(2\gamma+8)+a_1\right]^{\frac{1}{2}}\ .$$

The energy eigenvalance is given by (cf. eq (13))

$$E_\gamma=(\gamma+3)\frac{b}{\sqrt{2mc}}=(\gamma+3)\left[\sqrt{\frac{2c}{m^3}}(2\gamma+8)+\frac{2a}{m}\right]^{\frac{1}{2}}\ , \qquad (20)$$

where b is fixed from the following constraint

$$b=2\sqrt{c}\left[\sqrt{\frac{c}{2m}}(2\gamma+8)+a\right]^{\frac{1}{2}}\ . \qquad (21)$$

The normalized eigenfunctions are given by (cf.eq.(14))

$$\varphi_{\nu\gamma}(r) = N_\nu x^{\gamma+\frac{3}{2}} \exp\left[\frac{(-bx^2 - cx^4)}{4\sqrt{c}}\right] . \qquad (22)$$

Then from the transformation $\psi_{\nu\gamma}(x) = x^{-\frac{5}{2}}\varphi_{\nu\gamma}(x)$ and the constraint (21) reduces (22) and

$$\psi_{\nu\gamma}(x) = N_\gamma (x)^{\gamma-1} \exp\left\{-\left[\frac{1}{2}(c_1^{\frac{1}{2}}(2\gamma+8)+a_1)\right]^{\frac{1}{2}} x^2 - \frac{\sqrt{c_1}}{4}x^4\right\}. \qquad (23)$$

Also for $c = 0$ then from constraint (21) b=0, the potential in eq.(2) terns to the harmonic oscillator (h.o) potential $V(x) = (\alpha x^2 + \frac{\eta(\eta+1)}{x^2})$ with $\eta = \gamma + \frac{3}{2}$ then its exact energy spectra from equation (20) are given

$$E_{0,\gamma} = 2(\gamma+3)\sqrt{\frac{2a}{m}} = (\gamma+3)\sqrt{\frac{k}{m}} = 2(\gamma+3)\omega \ , \qquad (24)$$

where $k = 2a$ is the h.o potential strength and is a constant independent of N and the corresponding eigenfunctions are $f(x) = 1, N = 0$ .For ground state

$$\varphi_{0\gamma} = N_0 x^{\gamma+\frac{3}{2}} \exp(-\frac{m\omega}{2}x^2) \ . \qquad (25)$$

In term of relative particles position r_{12}, r_{23} and r_{31}

$$\varphi_{0\gamma}(r_{13}, r_{23}, r_{31}) = N_0\left[\frac{1}{3}(r_{12}^2 + r_{23}^2 + r_{31}^2)\right]^{\frac{\gamma}{2}+\frac{3}{4}} \exp\left[-\frac{m\omega}{6}(r_{12}^2 + r_{23}^2 + r_{31}^2)\right] , \qquad (26)$$

for the excited state $N \neq 0$ in this method we have

$$\varphi_N(x) = f_{N,\eta}(x) x^\eta \exp\left[-\frac{1}{2}\sqrt{a_1}x^2\right] = N_N F_{N,\eta} \exp(-\frac{m\omega}{2}x^2) \ . \qquad (27)$$

It is clear that $F_{N,\eta} = x^\eta \prod_{i=1}^{\eta}(x - \alpha_i^N)$ where $F_{0,\eta}(x) = N_0 x^{\gamma+\frac{3}{2}}$ the polynomial $F_{N,\eta}(x)$ is the spherical Hermite polynomial which shows our method is completely correct. With the normalization constant N_γ for eq (18) obtained from

$$\int_0^\infty |\psi_\gamma(x)|^2 x^3 dx = 1 \ , \qquad (28)$$

as [18]

$$N_\gamma = \sqrt{2}c^{\frac{(\gamma+3)}{8}} \exp(+\frac{b}{32c\sqrt{c}}) \left[\Gamma(\gamma+3) \times D_{-(\gamma+3)}(\frac{b}{\sqrt{2}}c^{\frac{3}{4}})\right]^{-\frac{1}{2}} , \qquad (29)$$

where $D_\nu(x)$ is the parabolic cylindrical function. Thus for the potential (11) the energy eigenvalues and the corresponding eigenfunctions are given by eqs.(13) and (20), respectively. It may be noted that from eq.(20) the ground state (zero-point) energy corresponding to $\gamma = 0$ is not zero but is given by $E_0 = \frac{3b}{\sqrt{2mc}} = 3\left[8\sqrt{\frac{2c}{m^3}} + \frac{2a}{m}\right]^{\frac{1}{2}}$ for $c = 0$ turns to the harmonic oscillator (h.o) ground state energy

Table1. The class hypercentral potentials $V(x)$ where $x^2 = \frac{1}{3}(r_{12}^2 + r_{23}^2 + r_{31}^2)$ for three body which allow us to obtain Schrodinger equation analytically with a suitable ansatz function.

Three body interacting potential $V(x)$	Ansatz function $g(x)$
$ax^{-2} + bx^2 + cx^4 + dx^6$	$\frac{1}{2}\alpha x^2 - \frac{1}{4}\beta x^4 + c\ln x$
$ax^{-2} + bx^{-1} + cx + dx$	$\frac{1}{2}\alpha x^2 + \beta$
$ax^{-4} + bx^{-3} + cx^{-2} + dx^{-1}$	$\frac{\alpha}{x}\beta x + \delta\ln x$
$ax^{-6} + bx^{-4} + cx^{-2} + dx^2$	$\frac{a}{x^2} + \frac{\beta}{x} + \gamma x + \delta\ln x$
$ax^2 + bx + \frac{c}{x}$	$\frac{1}{2}\alpha x^2 + \beta x$
$ax^2 + \frac{c}{x}$	$(1+\beta x^2)\exp(-\frac{1}{2}\alpha x^2)$
$ax^2 + bx^{-4} + cx^{-6}$	$\frac{1}{2}\alpha x^2 + \frac{1}{2}\beta x^{-2} + \delta\ln x$
$ax^2 - bx^4 + cx^6$	$\frac{1}{2}\alpha x^2 - \frac{1}{2}\beta x^4 + \delta\ln x$
$ax^{-2} + bx^{-\frac{3}{2}} + \frac{c}{x} + dx^{\frac{1}{2}}$	$\beta x^{\frac{1}{2}} + \delta\ln x$
$ax^{-2} + bx^{-\frac{4}{3}} + cx^{-\frac{2}{3}} + dx^{\frac{2}{3}}$	$\alpha x^{\frac{1}{3}} + \frac{3}{2}\beta x^{\frac{2}{3}}\delta\ln x$
$\frac{c}{x}$	$\beta x + \delta\ln x$
$\frac{D}{x^2} + \frac{c}{x}$	$\beta x + \delta\ln x$
$ax^2 - bx^4 + cx^6 - dx^8 + ex^{10}$	$\frac{1}{2}\beta x^2 - \frac{1}{4}\alpha x^4 + \frac{1}{6}\tau x^6 + \delta\ln x$

$E_0 = 3\sqrt{\frac{2a}{m}} = 3\sqrt{\frac{k}{m}} = 3\omega$. This shows that our results are correct. Furthermore the validity of the solution of the problem is limited by the constraint (19,21) on the parameter b. This parameter, in fact involves γ dependence through δ. The ansatz (15) can also be applied to further anharmonicity and the eigenvalues can be obtained as before (table 1). As an example let applied to the case when the potentials $V(x)$ involves further order a harmonicity.

However, in this case the normalization of the eigenfunctions becomes a difficult task. E.g. for the potential

$$V(x) = a_1 x^2 + b_1 x^4 + c_1 x^6 + d_1 x^8 + e_1 x^{10}\ , \qquad (30)$$

one can use $\varphi(r) = \exp\left[g(r)\right]$ with

$$g(x) = \frac{1}{2}\beta x^2 - \frac{1}{4}\alpha x^4 + \frac{1}{6}\tau x^6 + \delta \ln x\ . \qquad (31)$$

The expression for eigenvalues now becomes

$$\varepsilon_1 = -(2\gamma+6)\beta$$
$$(\beta = \frac{(4ce - d^2)}{4e\sqrt{e}}, \delta = 2\gamma + \frac{5}{2})\ , \qquad (32)$$

where various potentials parameters are re- defined in the spirit of eq (9) and now satisfy two constraints namely

$$\beta^2 - 2\alpha\delta - 3\alpha = a_1$$
$$2\alpha\beta - 2\tau\delta - 5\tau = -b_1\ . \qquad (33)$$

The eigenfunctions (not normalized) are given by

$$\varphi(x) \sim x^{\gamma+\frac{3}{2}} \exp\left[\frac{1}{2}\beta x^2 - \frac{1}{4}\alpha x^4 + \frac{1}{6}\tau x^6\right]. \qquad (34)$$

3. Conclusion

The analysis presented in this article has been carried assuming $S_3(or D_3)$ symmetry (i.e three identical constituents with identical interactions). The interaction between the three objects may be such that the geometric arrangement is that of an equilateral triangle with D_3 symmetry. The problem of the anharmonic oscillator with quartic type anharmonicity in the two body potential has been very widely studied [18] but in this article we have solved the radial Schrodinger wave equation rather exactly for the sextic three-body potential in three dimension (11) with the constraint (21) on the parameters. While eigenvalues and eigenfunctions for this potential are obtained in a closed form, the results are outlined for the potential (30). Furthermore the results obtained here seem to have some direct applications in fibre optics, where one solves [20] a similar problem of an inhomogeneous spherical -or circular wave guide with refractive index profile function of the type (11). Within this framework we can also study baryons to be built of three constituent quark parts. By making use of these methods we are able to calculate in a straight forward way all observable quantities and thus test various models. The fact that the formalism has been setup in a model-independent way as much as possible, gives the possibility to search for new physics. Such studies are in progress.

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