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Exponential law as a more compatible model to describe orbits of planetary systems

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Abstract
According to the Titus-Bode law, orbits of planets in the solar system obey a geometric progression. Many investigations have been launched to improve this law. In this paper, we apply square and exponential models to planets of solar system, moons of planets, and some extra solar systems, and compare them with each other.

Keywords: exponential and square models, planetary orbits, extra solar planets

1. Introduction

Human being has been attempting to find regularity in the universe. Explanation of planetary distances distribution is an example of these efforts. Titius (1772) and Bode (1776) proposed a law which describes the mean radii of planetary orbits in the following general form:

$$r_n = a + bc^n \,. \tag{1}$$

The constants a,b and c have no convincing physical meaning, neithey they have the empirical correlation with definite parameters for a given system. Therefore, this law has aroused many discussions [1]. Titius-Bode's relation embarrassingly breaks down for Neptune and objects farther than that, while it gives neat results for the eight first planets, including the asteroids. This is where people begin to doubt Titius-Bode's relation.

In recent years, some authors [2–5] have suggested a quantum-like approach to calculate the planetary orbits in our solar system, which has led to several impressive results.

Besides obtaining the observed orbits of all the planets and the asteroid belt, such models have made predictions, subsequently confirmed by observations. For example, the existence of asteroids orbiting between Uranus and Neptune, with orbit radius around 24.77 AU [6] was confirmed by recent discoveries [7]. Furthermore, the prediction of two intramercurial orbits with radii around 0.05 and 0.18 AU [8], absents of our solar system, has been verified in several extra-solar planetary systems during the last years [9].

In this paper the square and exponential models are applied to some extra solar planetary systems and then are compared with each other.

2. Bohr quantization model

The square model was created with supposition of quantization of orbital angular momentum (Bohr model of hydrogen atom). If the Planck's constant is expressed using the relationship,

$$h = 2\pi \frac{e^2}{\alpha_\varepsilon c} \,, \tag{2}$$

where $\alpha_\varepsilon \cong 2\pi \frac{1}{137}$ is the dimensionless fine structure constant, the Bohr-Sommerfeld rules can be written in the form

$$\oint p_j dq_j = n_j 2\pi \frac{e^2}{\alpha_\varepsilon c} \,. \tag{3}$$

One can imagine that, if the electron charge had been with higher precision in Planck's time, he would have introduced α_ε instead of h as a new constant. The form of the previous equation explicitly shows its 'atomic electrostatic' purpose. On the other hand, both the (atomic) electrostatic and the gravitational fields follow the r^2 law and only differ in their 'sources':

e^2 = proton charge times electron charge in the hydrogen atom,

GMm = mass m in the gravitational field, with a source of mass M.

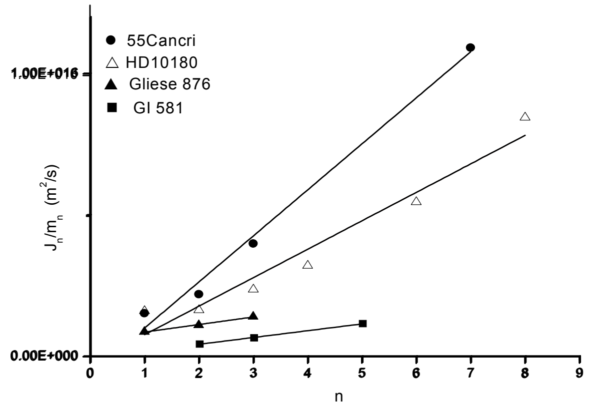

Figure 1. values of J/m in some extra solar system versus the number n.

For the first time, Bohr model was applied to solar system, moons of Jupiter, Saturn and Uranus [10]. In fact from Newton's law of gravity, assuming a circular orbit, one can obtain orbital radius as $r=\frac{(vr)^2}{GM}$, where G is gravitational constant and M is the mass of a central body. $vr=\frac{2\pi r^2}{T}$, angular momentum per unit mass of orbiting body, is proportional to n. So, $\frac{J}{m}=vr=Cn$. C value is nearly the same for all planets of a given system (see figure 1).

One can define the ratio $\frac{C}{M}$ as $\frac{C}{M}=fA$, where dimensionless factor f depends on particular system and A is a parameter with dimension of angular momentum per square mass and its value is $A=\frac{2\pi G}{\alpha c}=1.9157\times10^{-16}\,(Jskg^{-2})$, where c is the speed of light and α is fine–structure constant. So, discrete orbital radius of planets and moons is given by [10].

$$r_n=\frac{1}{G}(fA)^2Mn^2 . \tag{4}$$

Along with the orbital number n, an additional number k may be introduced as $nv_n=kv_0$, where v_0 is a fundamental velocity and k is a constant that shows the compression of orbits. So, eq. (4) is rewritten as

$$r_n=\frac{G}{v_0^2}M\frac{n^2}{k^2} . \tag{5}$$

Value of v_0 has been determined $v_0\approx 24\,\mathrm{km/s}$ for solar system (one of velocities of the quantized red shifts of galaxies). It is expected that we can apply $v_0\approx 24\,\mathrm{km/s}$ to systems with central star similar to the sun [11]. However, some authors prefer velocity of $v_0\approx 144\,\mathrm{km/s}$ for solar system [5]. Some other values of velocity increments of the quantized red shifts of galaxies are 18, 36, 72 and 144 km/s [12-14].

Table 1. Mass and semi-major of studied planetary systems (taken from http://exoplanet.eu/catalog-RV.php on 09February 2011).

SYSTEM	Name of Planet	M.Sin I (M_{Jup})	Semi-major (AU)
HD10180	HD10180 b *(unconfirmed)*	0.00424755	0.02225
	HD10180 c	0.041217	0.0641
	HD10180 d	0.03696945	0.1286
	HD10180 e	0.07897304	0.2699
	HD10180 f	0.07519743	0.4929
	HD10180 g	0.06733159	1.422
	HD10180 h	0.202624	3.4
55Cancri	55Cancri e	0.024	0.038
	55Cancri b	0.824	0.115
	55Cancri c	0.169	0.24
	55Cancri f	0.144	0.785
	55Cancri d	3.835	5.77
Kepler 11	Kepler 11 b	0.01353	0.091
	Kepler 11 c	0.0425	0.106
	Kepler 11 d	0.01919	0.159
	Kepler 11 e	0.02643	0.194
	Kepler 11 f	0.007237	0.25
	Kepler 11 g	0.95	0.462
Gliese 876	Gliese 876 d	0.021	0.02080665
	Gliese 876 c	0.7142	0.12959
	Gliese 876 b	2.2756	0.208317
	Gliese 876 e	0.046	0.3343
GI 581	GI 581 e	0.006104	0.03
	GI 581 b	0.0492	0.041
	GI 581 c	0.01686	0.07
	GI 581 d	0.02231	0.22
	GI 581 f *(unconfirmed)*	0.023	0.758
	GI 581 g *(unconfirmed)*	0.01	0.14601

2. 1. HD10180 System

This system has the most planets (6 confirmed planets and one unconfirmed planet) in extra solar planetary systems. The mass of central star in it is 1.06 $M_\odot$ and its spectral type is G1V. Most of the planets of the system have been discovered in 2010.

Table 1 shows the mass and semi-major axis of its planets which have been investigated.

Table 2. Obtained parameters of square law in studied systems.

System		Factor f	Orbital number	Number of unoccupied orbits	R-Square	Coefficient k
HD10180	First method	1.68 ± 0.64	1-9	6,8	0.6962	-
	Second method	2.29 ± 0.58	1-8	5,7	0.9194	6
55Cancri	First method	2.21 ± 0.31	1-11	5,6,7,8,9,10	0.9344	-
	Second method	3.49 ± 0.43	1-7	4,5,6	0.9941	4
Kepler11	First method	4.76 ± 0.77	1-6	1	0.9763	-
	Second method (Terrestrial planets)	5.26 ± 0.55	1-4	1	0.9956	9
	Second method (Giant planets)	5.76 ± 0.49	1-5	1,4	0.9771	8
Gliese 876	First method	4.65 ± 0.35	1-5	2	0.9489	-
	Second method	7.6 ± 0.15	1-5	1,2	0.9931	-

Except the terrestrial planet called HD10180 b, all other planets lie in giant planets group. Using the square model, orbit of planets is numbered in two ways. In the first method, all of the planets are considered in the form of a single system. Thus, the first to the fifth planets lie in n = 1-5 orbits, the sixth planet in n = 7 and the seventh planet in n =9, which means that n = 6 and n = 8 orbits are unoccupied.

In the second method, no terrestrial planet is considered. In other words, giant planets are investigated apart from terrestrial planets. Thus, the first to the fourth giant planets lie in n = 1-4 orbits and the fifth planet lies in n = 6 orbit and the last one in n = 8 orbit. This means that n = 5 and n = 7 orbits are unoccupied and there is an expectation of discovering two other giant planets in the future. If terrestrial planet is labeled with n =1 then f = 1.67. Of course until there's no discovery of more terrestrial planets in this system, one can't discuss the numbering of this planet firmly. Some of parameters that have been obtained from these investigations are shown in table 2.

With regard to f value in the first method, radius of n = 1 orbit is estimated about 0.0219 AU from eq. (4), which is the same as unconfirmed planet. So existence of a planet in such an orbit is possible. In the second method, with the use of f value and eq. (4), unoccupied n = 5 orbit's radius is estimated about 1.01 AU, which is approximately the same as radius which has been recently predicted by the use of pair-correlation analysis of HD10180 (0.92 AU). The correlations between the positions of the exoplanets are determined from the pair-correlation function, which reads

$$P(\Delta) = \int \rho(x+\Delta)\rho(x)dx\,,$$

here $\rho(x)$ is a distribution that measures the positions, x_i, of the planets on a logarithmic scale [15].

2. 2. 55Cancri system

As shown in table 1, this system has 5 planets (55Cancri e is considered as terrestrial planet). Its central star's mass is about 1.03 $M_\odot$ and its spectral type is G8V. This system is studied in two methods just like HD10180. In the first method, all of the planets are considered in the form of a single system. So, the first to the fourth planets lie in n = 1 to n = 4 and the fifth planet in n = 11. Thus 6 unoccupied orbits (n = 5-10) are produced in it. In the second method, giant and terrestrial planets are studied separately. So, the first to the third planets lie in n = 1 to n = 3, the fourth giant planet in n = 7. As a result n = 4, 5, 6 orbits are unoccupied. According to the long distance between the fourth and the fifth planets (0.781-5.77 AU), it is expected that new planets will be discovered in near future between these two. Like the former system there is only one terrestrial planet in 55cancri. So, numbering of terrestrial planets is avoided, since the numbering in this method is uncertain. Because of engendering a lot of empty orbits in the first method and with regard to values of R-Square, it is preferred that in the study of this system, giant and terrestrial planets are considered as two separate systems. Results of this study are shown in table 2.

2. 3. Kepller11 system

This system has 6 planets. It is a little different from former systems, because giant planets lie between terrestrial planets and they are not completely separated. Mass of central star in it is 0.95 $M_\odot$. If square law is applied on this system in such a way that all planets lie in a group, orbital number of n = 2 would be assigned to both Kepler11 b and Kepler11 c planets, and n =1 orbit is predicted unoccupied; but according to mass of central star, existence of a planet in such a small orbit seems approximately impossible. On the other hand, if terrestrial and giant planets of this system are studied discretely, in each group 1 and 2 unoccupied orbits are predicted respectively. R-Square values of this system also show that if giant and terrestrial planets are discretely studied, there will be better fit between the model and observation data (Table 2).

2. 4. Gliese876 system

This system has four planets. The low mass and probable

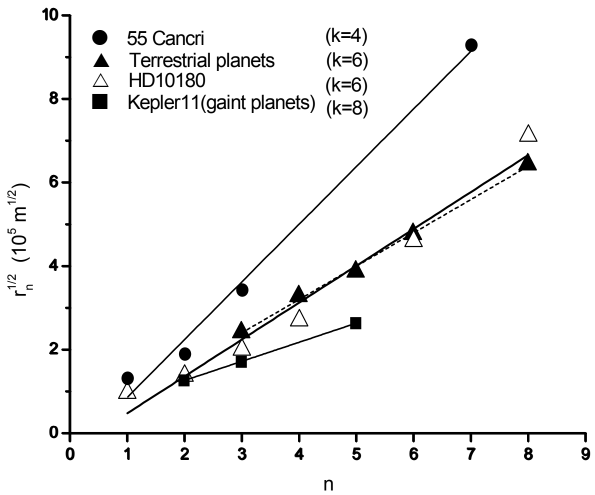

Figure 2. Correlation of the square root of the semi-major axes with the orbital numbers n for some extra-solar planetary system.

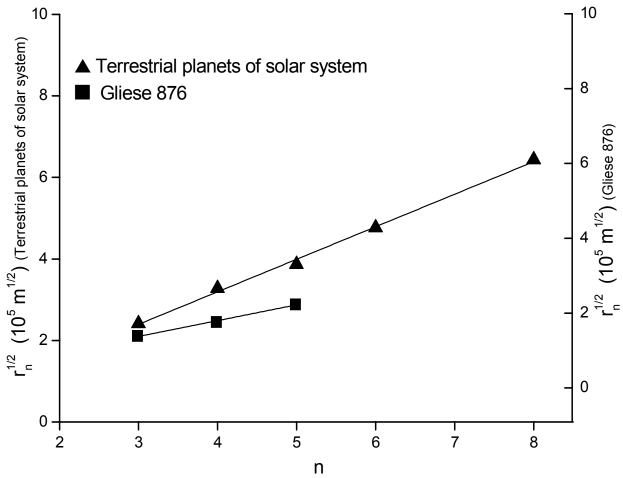

Figure 3. Correlation of the square root of the semi-major axes with the orbital numbers n for terrestrial planets of solar system and Gliese 876 system.

temperature of the Gliese 876 d planet have led to suggestions that it may be a terrestrial planet. This type of massive terrestrial planet could be formed in the inner part of the Gliese 876 system from material pushed towards the star by the inward migration of the gas giants. Like former systems, results of applying the square law are shown in table 2. If giant and terrestrial planets are discretely studied in this system, square law has better fit to observational data.

2. 5. GI581 system

This system has four planets. Except GI581 e and GI581 b, other planets of this system lie in the category of Super-Earth planets. Because of the unknown radius of these planets, one cannot firmly discrete giant and terrestrial planets in this system. So in this paper, we avoid applying square law on this system.

If $v_0 \approx 24\,\mathrm{km/s}$ is chosen with regard to A. Rubcic and J. Rubcic [11], because $nv_n \approx 159\,\mathrm{km/s}$ is for Jovian planets of HD10180 system, k coefficient is 6; this value is greater than k=1 for Jovian planets of solar system. This shows that compression of orbits of Jovian planets in HD10180 system is more than solar system's Jovian planets and is very similar to solar system's terrestrial planets (k=6). Also, for other systems k values are determined in this way. In figure 2, one can compare compression of orbits of some extra solar planetary systems with compression of orbits of solar system's terrestrial planets.

Regarding the Gliese876 system whose central star`s mass is 0.33 $M_\odot$, k=6 is obtained. But as it is shown in figure 3, compression of orbits in this system is not similar to terrestrial planets of solar system so much as HD10180 system (slopes of lines are not equal). So it can be firmly stated that speed of $v_0 \approx 24\,\mathrm{km/s}$ and thus k=6 for Gliese876 is not a good choice. Using Bohr model to describe orbits distribution of the solar system and some extra solar planetary systems, has shown that this model can describe distribution of planetary orbits well. Using this model is more useful when Jovian and terrestrial planets are considered separately.

Also, this study confirms this hypothesis that in planetary systems with similar central star, value of peculiar velocity v_0 is the same.

3. Exponential model

One of the modified forms of Titius-Bode law is its exponential form. This form of Titius-Bode law is very interesting, because if it is true, an important step to prove the physical nature of this controversial law would be taken. Some authors suppose exponential form as $r_n = c_1 c_2^n$ [16]. But in this paper, exponential form of $r_n = ae^{bn}$ is considered. For the Solar System we have b=0.53707, a=0.21363 AU, and e^b=1.7301 .

The parameter α has the dimension of a length and it is linked to the radius of the first orbit of the system considered (in fact $r_1 = ae^b$). If we quantize the angular momentum *per unit mass* (i.e. the velocity field instead of the momentum field)

$$\begin{cases} m\dfrac{v^2}{r} = \dfrac{GMm}{r^2}, \\ \dfrac{J}{m} = vr = qe^{bn/2}, \end{cases} \tag{6}$$

where $q = \sqrt{GMa}$. So

$$\begin{cases} v_n = \dfrac{GM}{q}\exp\left(-\dfrac{bn}{2}\right) = v_1 \exp\left(-\dfrac{b(n-1)}{2}\right), \\ r_n = \dfrac{q^2}{GM}\exp(bn) = v_1 \exp(b(n-1)). \end{cases} \tag{7}$$

Substituting this equation in Kepler third law:

Table 3. Results of applying exponential and square models to solar system and moons of planetary system.

System	Square model			Exponential model		
	Orbital number	Number of unoccupied orbitals	R-Square	Orbital number	Number of unoccupied orbitals	R-Square
Terrestrial planets	n=1-6	n=1,2	0.9976	n=1-9	n=5	0.9946
Jovian planets	n=1-6	-	0.9933			
Moons of Jupiter	n=1-6	n=1	0.9777	n=1-8	-	0.9947
Moons of Saturn	n=1-19	n=1-5 n=12-18	0.9045	n=1-9	-	0.9882
Moons of Uranus	n=1-8	n=1,2	0.8280	n=1-6	-	0.9925

Table 4. Orbital distance and orbital number of moons of Jupiter and Saturn systems.

Moons of Jupiter			Moons of Saturn		
Name	Distance(AU)	Orbital number	Name	Distance(AU)	Orbital number
Metis	0.000856	n=2	Prometheus	0.00093182	n=6
Andrastea	0.000863	n=2	Epimetheus	0.00101203	n=6
Amalthea	0.001213	n=2	Janus	0.00101203	n=6
Thebe	0.001484	n=2	Mimas	0.00123997	n=7
Io	0.002821	n=3	Enceladus	0.00159091	n=8
Europa	0.004489	n=4	Tethys	0.00196992	n=9
Ganymede	0.00716	n=5	Dione	0.00252273	n=10
Callisto	0.012593	n=6	Rhea	0.00352273	n=11

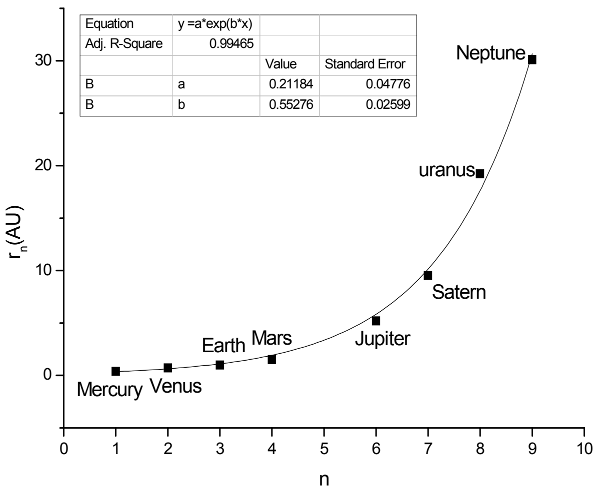

Figure 4. Fit of exponential model on distance of planets of solar system.

$$T = \frac{2\pi q^2}{(GM)^2} e^{3bn/2} . \qquad (8)$$

Two important questions are raised in relation to Titius-Bode law:

Is there a simple mathematical relation between distances of planets from central star? And the other question is whether such a relation has a physical base? These subjects are discussed to see whether the Titius-Bode law is valid as a universal model in extra solar systems or not.

In using this model, one obtains a better fit if all planets lie in a group. Table 3 shows R-Square values that result from fitting of exponential and Bohr models to observational data.

From table 3 following notifications are obtained:

1. Not only exponential model has a good fitting to observational data, but also many unoccupied orbits are not generated in it, which is another advantage of this model. Notice that, as shown in figure 4, unoccupied orbit n=5 in the solar system is correspondent to asteroid belt.
2. In the old form of Titius-Bode law, orbital number belongs to mercury $n=-\infty$ that has no physical concept. But using the exponential form, it is not necessary for mercury to be excluded.
3. It is important to note that in square law, for better fitting, some of the moons of Jupiter and Saturn (Table 4) as well as Kepler11 system, are given a repetitive number [10], while there is no such problem in exponential model.

As in figure 5, period distribution of solar system planets obey 527882 $e^{0.78347n}$; these coefficients are almost near to the obtained coefficients from eq.(8). Hence exponential model can be applied to the Kepler law, too.

3. 1. HD10180 and 55Cancri systems

In figure 6 we fit the exponential model on planets distribution in HD10180 and 55Cancri systems. Exponential model doesn`t predict any unoccupied orbit between discovered planets of HD10180 system and it is expected that if a planet is to be discovered in this system, it would lie in n=8 orbit and on 42AU distance; which has a little probability. In the case of 55Cancri system, there have recently been some efforts to fit the old forms of Titius-Bode law on observational data. It is concluded that this law cannot describe planetary orbits distribution in this system [17]. But as shown in figure 6, exponential form of this law with prediction of

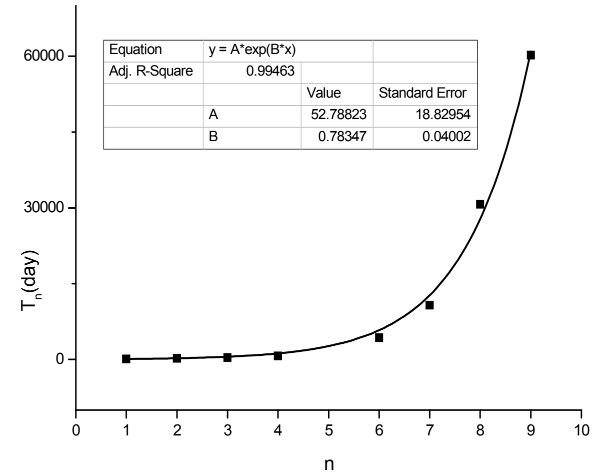

Figure 5. Fit of exponential model on period of planets of solar system.

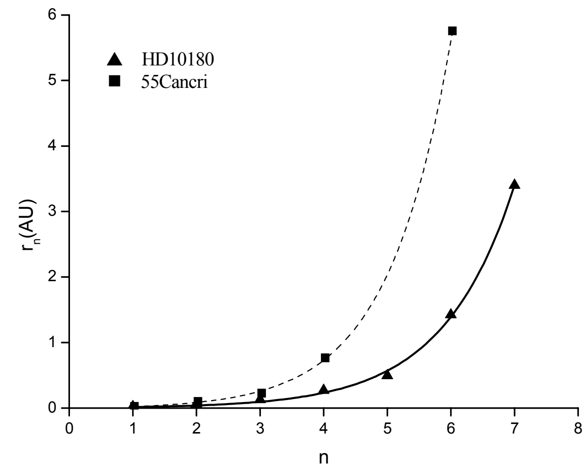

Figure 6. Fit of exponential model on HD10180 and 55Cancri systems.

Table 5. Results of applying exponential model on Kepler11 and Gliese 876 and GI 581.

System	Orbital number	Number of unoccupied orbits	R-Square
Kepler11	n=1-9	n=3, 7, 8	0.9990
Gliese 876	n=1-6	n=2, 3	0.9975
GI 581	n=1-6	n=4,5	0.9983

Table 6. Obtained a and b Coefficient in studied systems.

	Solar system	HD10180	55Cancri	Kepler11	Gliese 876	GI 581
a	0.2118	0.0065	0.0133	0.0555	0.0202	0.0179
b	0.5597	0.8944	1.0109	0.2627	0.3985	0.4885
$M(M_{sun})$	1	1.06	1.03	0.95	0.31	0.33

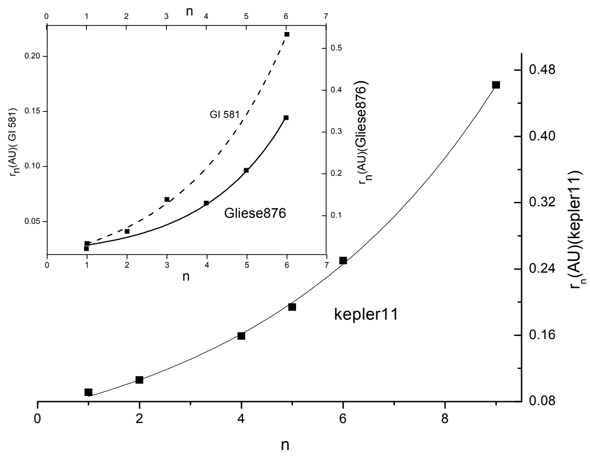

Figure 7. Fit of exponential model on Keplerr11 and Gliese876 and GI581 systems.

unoccupied n=5 orbit (r=2.07 AU), fits well with this system (with regard to [18]). It is interesting that this unoccupied orbit has been predicted with the use of square model on this system (r=2.12 AU, n=4).

3. 2. Keplerr11 and Gliese876 and GI581 systems

Further, the other three extra solar planetary systems, Keplerr11, Gliese876 and GI581, have been studied with exponential model. As in figure 7, this study shows that some unoccupied orbits are engendered in each system. Table 5 shows the results of this study.

Two planets are recently discovered on radial distances 0.14AU and 0.75AU in GI581 system. Although these discoveries have not been confirmed yet, it is interesting that these planetary distances respectively corresponds with orbits n=4 and n=9 with radiuses of about 0.1AU and 0.71AU, which had been predicted as being unoccupied by exponential model.

In fact, this model predicts undiscovered planets in extra solar planetary systems. It is clear that if planetary distances distribution is compatible to this law not only in solar system, but also in other planetary systems, it is shown that this model is more than a numerical correspondence.

The above study reveals that there is a good correspondence between this model and extra solar planetary systems. The obtained values of a and b coefficients for each system and central star mass are shown in table 6.

One can see that b coefficient in systems with more massive central star is greater than those with lower central star mass. Thus it is possible that there is a physical relation between b coefficient and central star mass.

4. Conclusions

1. Application of square model to describe the orbit distribution of extrasolar planetary systems is more useful when terrestrial and Jovian planets are considered as separated systems.
2. Also, this study confirms this hypothesis that in planetary systems with similar central star, the value of peculiar velocity v_0 is the same.
3. Coincidence of exponential model to solar system, moons of Jupiter and Saturn and some extra solar planetary systems concludes that this model corresponds better to the distribution of orbits with prediction of some empty orbits in these systems, although it is necessary to make more observations on these systems in future to confirm this hypothesis.
4. Comparing two square and exponential models, it is suggested that exponential model has a better coincidence to observational data.

It is needed to separate terrestrial and Jovian planets system in square model, while there is no such a need in exponential model. In square model, more numbers of empty orbits are predicted than exponential model. Also, in square model for a better correspondence, sometimes it is necessary that common orbital numbers are given to some planets, but this does not necessarily exist in exponential model.

5. This investigation shows that in exponential model there can be a physical relation between b parameter and central star mass of a planetary system. If this relation is detected clearly, this law would be accepted as a universal model to describe the planetary system orbits distribution. So, the obtained results in calculating planetary orbits in the present work, as well as those obtained by other authors, are enough to encourage further studies.

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