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Research note

Bulk viscous string cosmological models in Saez – Ballester theory of gravitation

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Abstract
Spatially homogeneous Bianchi type-II, VIII and IX anisotropic, as well as isotropic cosmological models can be obtained in a scalar tensor theory of gravitation proposed by Saez and Ballester (1986) when the source for energy momentum tensor is a bulk viscous fluid containing one-dimensional cosmic strings. All the models obtained and presented here are expanding, non-rotating and accelerating. Also, some important features of the models thus obtained have been discussed.

Keywords: Bianchi type - II, VIII and IX metrics, Bulk viscosity, Cosmic strings, Saez – Ballester theory

1. Introduction

Spatially homogeneous and anisotropic cosmological models play a significant role in the description of the large-scale behavior of the Universe. In search for a realistic picture of the early Universe, such models have been widely studied within a framework of General Relativity. In order to study the evolution of the universe, many authors have constructed cosmological models containing the viscous fluid. Cosmological models with bulk-viscosity are important since bulk-viscosity plays a greater role in getting the accelerated expansion of the universe popularly known as the inflationary phase. The role of bulk viscosity in the cosmic evolution, especially at its early stages, seems to be significant. The distribution of matter can be satisfactorily described by a perfect fluid due to the large-scale distribution of galaxies in our universe. However, the observed physical phenomena such as the large entropy per baryon and the remarkable degree of isotropy of the cosmic microwave background radiation suggest the analysis of dissipative effects in cosmology. Furthermore, there are several processes which are expected to give rise to viscous effects. These are the decoupling of neutrinos during the radiation era and the decoupling of radiation and matter during the recombination era. Bulk viscosity is associated with the GUT phase transition and string creation. The presence of viscosity in the fluid gives many interesting features in the dynamics of homogeneous cosmological models. The possibility of bulk viscosity leading to inflationary like solutions in general relativistic FRW models has been discussed by several authors. Barrow (1), Padmanabhan and Chitre (2), Pavon *et al*. (3), Martens, (4), Lima *et al*. (5)]. Wang (6, 7, 8), Bali and Dave (9), Bali and Pradhan (10), Tripathy *et al*. (11), Tripathy *et al*. (12) and Rao *et al*. (13) have studied various Bianchi type cosmological models in the presence of cosmic strings and bulk viscosity. Rao and Sireesha (14, 15) have also discussed the axially symmetric string cosmological model with bulk viscosity in the self-creation theory of gravitation.

The study of string theory has received considerable attention in cosmology. String cosmological models have attracted more and more attention since cosmic strings are important in the early stages of the evaluation of the universe before the practical creation. Spontaneous symmetry breaking in elementary practical physics has given rise to topological defects known as cosmic strings. The gravitational effects of such objects are of practical interest since they are considered as possible seeds for galaxy formation and gravitational lenses. In this regard, Rao *et al*. [16, 35, 36] have obtained Bianchi type-II,VIII and IX string cosmological models in Brans-Dicke theory of gravitation. Rao *et al.* (34) have also studied axially symmetric string cosmological models in the Brans-Dicke theory of gravitation.

Saez and Ballester (17) formulated a scalar-tensor theory of gravitation in which the metric was coupled with a dimensionless scalar field. This coupling led to a satisfactory description of the weak fields. In spite of the dimensionless character of the scalar field, an antigravity regime appears. This theory also suggests a possible way to solve themissing matter problem in nonflat FRW cosmologies.

We assume the lagrangian

$$L = R - W\phi^{n}\phi_{,\gamma}\phi^{,\gamma}\,, \tag{1}$$

R is the scalar curvature, ϕ is the dimensionless scalar field, W and n are arbitrary dimensionless constants, and $\phi^{,\gamma}$ is the contraction $\phi_{,\alpha}g^{\alpha\gamma}$ (the partial derivatives are denoted by a comma and the covariant derivatives by a semicolon in the usual way).

For a scalar field having the dimensions of G^{-1} , the lagrangian (1.1) is not physically admissible because the two terms of the right-hand side of eq. (1.1) have different dimensions. However, it is a suitable lagrangian in the case of a dimensionless scalar field.

From the above lagrangian, we can build the action

$$I = \int_{\Sigma} (L + \chi L_m)(-g)^{1/2}\, dX^1 dX^2 dX^3 dX^4 \tag{2}$$

where L_m is the matter lagrangian, g is the determinant of the matrix g_{ij}, X^i are the coordinates, Σ is an arbitrary region of integration, and $\chi = 8\pi$ (we use the geometrized units).

By considering arbitrary independent variations of the metric and the scalar field vanishing at the boundary of Σ , the variational principle

$$\delta I = 0 \tag{3}$$

leads to the field equations given by Saez- Ballester (1986) for the combined scalar and tensor fields (using the geometrized units with $c = 1$, $8\pi G = 1$) are

$$G_{ij} - \omega\varphi^{m}\left(\varphi_{,i}\varphi_{,j} - \frac{1}{2}g_{ij}\varphi_{,k}\varphi^{,k}\right) = -T_{ij} \tag{4}$$

and the scalar field ϕ satisfies the equation

$$2\varphi^{m}\varphi^{i}{}_{;i} + m\varphi^{m-1}\varphi_{,k}\varphi^{,k} = 0 \tag{5}$$

where $G_{ij} = R_{ij} - \frac{1}{2}R\,g_{ij}$ is an Einstein tensor, R the scalar curvature, ω and m are constants, and T_{ij} is the stress energy tensor of the matter.

Also, the energy conservation equation is given by

$$T^{ij}{}_{;j} = 0 \tag{6}$$

The study of cosmological models in the framework of scalar–tensor theories has been the active area of research for the last few decades. In particular, Rao *et al.* (14, 15 a, b) have investigated several aspects of the cosmological models in Saez-Ballester (17) scalar-tensor theory. Rao *et al.* (18) have also discussed Bianchi type -II, VIII and IX perfect fluid dark energy cosmological models in the Saez - Ballester and general theory of gravitation. Also, Rao *et al.* (17) have obtained perfect fluid dark energy cosmological models in this theory and the Einstein's theory of gravitation.

Bianchi type space- times play a vital role in understanding and describing he early stages of evolution of the universe. In particular, the study of Bianchi types II, VIII and IX universes can be important because familiar solutions like FRW universe with a positive curvature, the de Sitter universe, the Taub- Nut solutions etc. correspond to Bianchi type II, VIII and IX space- times. Shanthi and Rao (21) studied Bianchi types VIII and IX models in Lyttleton-Bondi Universe. Also, Rao and Sanyasi Raju, (19) and Sanyasi Raju and Rao, (20) have studied Bianchi types VIII and IX models in Zero mass scalar fields and self creation cosmology. Reddy *et al.* (22), for instance, studied Bianchi types II, VIII and IX models in the scale covariant theory of gravitation. Raj Bali and Dave (23), Raj Bali and Yadav (24) studied Bianchi type IX string as well as viscous fluid models in general relativity. Rahaman *et al.* (25) have investigated Bianchi type- IX string cosmological model in a theory of gravitation formulated by Sen (26) based on the Lyra (27) manifold. Naidu *et al.* (28) have also discussed LRS Bianchi type-II universe with cosmic strings and bulk viscosity in a scalar tensor theory of gravitation using the negative constant deceleration parameter proposed by Bermann (29). Rao *et al.* (18) have further studied Bianchi types II, VIII and IX string cosmological models with bulk viscosity in a theory of gravitation. Rao and Sireesha (14 a) have been concerned with Bianchi types II, VIII and IX string cosmological models with bulk viscosity in the Brans-Dicke theory of gravitation. Rao *et al.* (30) have obtained Bianchi type-I string cosmological models with bulk viscosity in the Bimetric theory of gravitation. Rao and Neelima have also investigated string cosmological models with bulk viscosity in the Nordtvedt's general scalar tensor theory of gravitation [37]. Rao *et al.* (31) have discussed the Perfect fluid cosmological models in a modified theory of gravity. Recently, Rao *et al.* (32) have studied bulk viscous cosmological models coupled with a scalar field in the Einstein theory and the f (R,T) theory of gravity, respectively.

In this paper, we will discuss spatially homogeneous Bianchi type-II, VIII and IX anisotropic as well as isotropic bulk viscous string cosmological models, without using the negative constant deceleration parameter proposed by Bermann (29), in the Saez - Ballester (17) theory of gravitation.

2. Metric and Energy Momentum Tensor:

We consider a spatially homogeneous Bianchi type-II, VIII and IX metrics of the form

$$ds^2 = dt^2 - R^2\left[d\theta^2 + f^2(\theta)d\phi^2\right] - S^2\left[d\psi + h(\theta)d\phi\right]^2 \tag{7}$$

where (θ,ϕ,ψ) are the Eulerian angles, Rand S are the

functions of t only.
It represents
Bianchi type- II if $f(\theta)=1$ and $h(\theta)=\theta$ Bianchi type - VIII if $f(\theta)=Cosh\theta$ and $h(\theta)=Sinh\theta$
Bianchi type - IX if $f(\theta)=\sin\theta$ and $h(\theta)=\cos\theta$
The energy momentum tensor for a bulk viscous fluid containing one dimensional string is given by

$$T_{ij}=(\rho+\overline{p})u_i u_j-\overline{p}g_{ij}-\lambda x_i x_j \tag{8}$$

and

$$\overline{p}=p-3\xi H\ ,\ p=\epsilon_0\ \rho\quad(0\le\epsilon_0\le 1) \tag{9}$$

where $\overline{p}$ is the total pressure which includes the proper pressure, ρ is the rest energy density of the system , λ is the tension in the string , $\xi(t)$ is the coefficient of bulk viscosity, $3\xi H$ is usually known as bulk viscous pressure, H is the Hubble parameter, u^i is the four velocity vector, and x^i is a space-like vector which represents the anisotropic directions of the string.
Here, u^i and x^i satisfy the equations

$$g_{ij}u^i u^j=-1\,,$$

$$g_{ij}x^i x^j=1\,,$$

and

$$u^i x_i=0\,. \tag{10}$$

We assume that the string is lying along the z-axis. The one dimensional strings are assumed to be loaded with particles and the particle energy density is $\rho_p=\rho-\lambda$.
In a commoving coordinate system, we get

$$T_1^1=T_2^2=-\overline{p},\ T_3^3=\lambda-\overline{p},\ T_4^4=\rho \tag{11}$$

where $\rho,\lambda,\overline{p}$ and ϕ are the functions of time 't' only.

3. Solutions of Field equations:
Now, with the help of (8) to (11), the field equations (1) for the metric (2.1) can be written as

$$\frac{\ddot{R}}{R}+\frac{\ddot{S}}{S}+\frac{\dot{R}\dot{S}}{RS}+\frac{S^2}{4R^4}-\frac{\omega}{2}\varphi^m\dot{\varphi}^2=-\overline{p} \tag{12}$$

$$2\frac{\ddot{R}}{R}+\frac{\dot{R}^2+\delta}{R^2}-\frac{3S^2}{4R^4}-\frac{\omega}{2}\varphi^m\dot{\varphi}^2=\lambda-\overline{p} \tag{13}$$

$$2\frac{\dot{R}\dot{S}}{RS}+\frac{\dot{R}^2+\delta}{R^2}-\frac{S^2}{4R^4}+\frac{\omega}{2}\varphi^m\dot{\varphi}^2=\rho \tag{14}$$

$$\ddot{\varphi}+\dot{\varphi}\left(2\frac{\dot{R}}{R}+\frac{\dot{s}}{S}\right)+\frac{m}{2\varphi}\dot{\varphi}^2=0 \tag{15}$$

$$\dot{\rho}+(\rho+\overline{p})(2\frac{\dot{R}}{R}+\frac{\dot{S}}{S})-\lambda\frac{\dot{S}}{S}=0\ . \tag{16}$$

Here, the over head dot denotes differentiation with respect to't'.

When $\delta=0,\text{-}1\,\&+1$, the field equations (12) to (16) correspond to the Bianchi types II, VIII and IX universes, respectively.

By using the transformation $dt=R^2SdT$, the above field equations (12) to (16) will be reduced to

$$\left(\frac{R'}{R}\right)'+\left(\frac{S'}{S}\right)'-\left(\frac{R'}{R}\right)^2-2\left(\frac{R'S'}{RS}\right)+\frac{S^4}{4}-\frac{\omega}{2}\varphi^m\varphi'^2=-\overline{p}R^4S^2 \tag{17}$$

$$2\left(\frac{R'}{R}\right)'-\left(\frac{R'}{R}\right)^2-2\left(\frac{R'S'}{RS}\right)-\frac{3S^4}{4}+\delta R^2S^2-\frac{\omega}{2}\varphi^m\varphi'^2=(\lambda-\overline{p})R^4S^2 \tag{18}$$

$$\left(\frac{R'}{R}\right)^2+2\left(\frac{R'S'}{RS}\right)-\frac{S^4}{4}+\delta R^2S^2+\frac{\omega}{2}\varphi^m\varphi'^2=\rho R^4S^2 \tag{19}$$

$$\varphi''+\frac{m}{2\varphi}\varphi'^2=0 \tag{20}$$

$$\rho'+(\rho+\overline{p})(2\frac{R'}{R}+\frac{S'}{S})-\lambda\frac{S'}{S}=0 \tag{21}$$

Here, the over head dash denotes differentiation with respect to 'T'.

The field equations (17) to (21) are only four independent equations with seven unknowns: $R,S,\rho,\lambda,\overline{p}\,\&\,\varphi$, which are the functions of 'T'. Since these equations are non-linear in nature, in order to get a deterministic solution, we take the following plausible physical conditions:

(1). The shear scalar σ is proportional to the scalar expansion, θ, so that we can take a linear relationship between the metric potentials R and S , i.e.,

$$R=S^n \tag{22}$$

where n is an arbitrary constant.

(2). A more general relationship between the proper rest energy density, ρ , and the string tension density, λ , is taken to be

$$\rho=r\lambda \tag{23}$$

where r is an arbitrary constant which can take both positive and negative values. The negative value of r leads to the absence of strings in the universe and the positive value shows the presence of one dimensional string in the cosmic fluid. The energy density of the particles attached to the strings is

$$\rho_p=\rho-\lambda=(r-1)\lambda \tag{24}$$

From equations (17) - (24), we get

$$\frac{S''}{S}r(n-1)-\frac{S'^2}{S^2}[r(n-1)+n(n+2)]-\frac{(4r-1)}{4}S^4+\delta(r-1)S^{2n+2}-\frac{\omega}{2}\varphi^m\varphi'^2=0 \tag{25}$$

From equation (20), we get

$$\varphi=\left[\left(\frac{m+2}{2}\right)(k_1T+k_2)\right]^{\frac{2}{m+2}}\ ,\ m\neq-2 \tag{26}$$

Bianchi type-II ($\delta=0$) cosmological model:
If $\delta=0$, from equation (25), we get

$$C_1\frac{S''}{S}-C_2\frac{S'^2}{S^2}-\frac{(4r-1)}{4}S^4-\frac{\omega}{2}k_1^{\,2}=0 \tag{27}$$

where $C_1=r(n-1)$,

$C_2=r(n-1)+n(n+2)$.

With a suitable substitution, the equation (27) can be written as

$$S'^2=W^2S^6-\gamma^2S^2 \tag{28}$$

where $W^2=\frac{(4r-1)}{4[2r(n-1)-n(n+2)]}$ and $\gamma^2=\frac{\omega k_1^{\,2}}{2n(n+2)}$.

From equation (28), we get

$$S^2=\left(\frac{\gamma}{W}\right)\sec 2\gamma T \tag{29}$$

From equations (22) and (29), we get

$$R^2=\left[\left(\frac{\gamma}{W}\right)\sec 2\gamma T\right]^n \tag{30}$$

From equations (19), (26), (29) and (30), we get the energy density

$$\rho=\frac{n(n+2)W^{(2n+1)}}{\gamma^{(2n-1)}}\tan^2 2\gamma T\,(\sec 2\gamma T)^{-(2n+1)}-\frac{1}{4}\left(\frac{\gamma}{W}\right)^{1-2n}(\sec 2\gamma T)^{1-2n}+\frac{\omega}{2}k_1^{\,2}\left(\frac{\gamma}{W}\right)^{-(2n+1)}(\sec 2\gamma T)^{-(2n+1)} \tag{31}$$

From equations (17), (26), (29) and (30), we get the total pressure

$$\bar{p}=\left(\frac{\gamma}{W}\right)^{-(2n+1)}(\sec 2\gamma T)^{-(2n+1)}\left[n(n+2)\gamma^2\tan^2 2\gamma T+\frac{\omega}{2}k_1^{\,2}\right]-(\sec 2\gamma T)^{1-2n}\left[2(n+1)\frac{W^{(2n+1)}}{\gamma^{(2n-1)}}+\frac{1}{4}\left(\frac{\gamma}{W}\right)^{1-2n}\right] \tag{32}$$

The proper pressure is given by

$$p=\epsilon_0\,\rho=\frac{n(n+2)W^{(2n+1)}\epsilon_0}{\gamma^{(2n-1)}}\tan^2 2\gamma T\,(\sec 2\gamma T)^{-(2n+1)}-\frac{1}{4}\left(\frac{\gamma}{W}\right)^{1-2n}\epsilon_0\,(\sec 2\gamma T)^{1-2n}+\frac{\omega}{2}k_1^{\,2}\epsilon_0\left(\frac{\gamma}{W}\right)^{-(2n+1)}(\sec 2\gamma T)^{-(2n+1)} \tag{33}$$

From equations (18), (26), (29), (30) and (32), we get the string tension density

$$\lambda=\left(\frac{\gamma}{W}\right)^{-(2n+1)}(\sec 2\gamma T)^{1-2n}\left[2(n-1)\gamma^2-\left(\frac{\gamma}{W}\right)^2\right] \tag{34}$$

The particle energy density is given by

$$\rho_p=\rho-\lambda=\left(\frac{\gamma}{W}\right)^{-(2n+1)}(\sec 2\gamma T)^{1-2n}(r-1)\left[2(n-1)\gamma^2-\left(\frac{\gamma}{W}\right)^2\right] \tag{35}$$

The coefficient of bulk viscosity is given by

$$\xi=\left(\frac{\gamma}{W}\right)^{-(2n+1)}\frac{(1-\epsilon_0)}{(2n+1)\gamma}\left[n(n+2)\gamma^2\tan^2 2\gamma T+\frac{\omega}{2}k_1^{\,2}\right](\sec 2\gamma T)^{-(2n+1)}\cot 2\gamma T+\frac{1}{4}\left(\frac{\gamma}{W}\right)^{1-2n}\frac{(\epsilon_0-1)}{(2n+1)\gamma}(\sec 2\gamma T)^{1-2n}\cot 2\gamma T-\left(\frac{\gamma}{W}\right)^{-(2n+1)}\frac{2(n+1)\gamma}{(2n+1)}(\sec 2\gamma T)^{1-2n}\cot 2\gamma T \tag{36}$$

The generalized mean Hubble parameter (H) is

$$H=\frac{1}{3}\left(H_1+H_2+H_3\right)=\frac{-(2n+1)\gamma}{3}\tan 2\gamma T \tag{37}$$

The metric (2.1), in this case, can be written as

$$ds^2=\left[\left(\frac{\gamma}{W}\right)\sec 2\gamma T\right]^{\frac{2n+1}{2}}dT^2-\left[\left(\frac{\gamma}{W}\right)\sec 2\gamma T\right]^n(d\theta^2+d\phi^2)-\left[\left(\frac{\gamma}{W}\right)\sec 2\gamma T\right](d\psi+\theta d\phi)^2 \tag{38}$$

Thus, (38), together with (26), (31), (32), (34) and (36), constitutes a Bianchi type-II string cosmological model with bulk viscosity in Saez - Ballester (17) theory of gravitation. This model is entirely different from the model obtained by Naidu *et al*. (28), since we have obtained it without using the negative constant deceleration parameter proposed by Bermann (29).

Bianchi type-VIII ($\delta=-1$) cosmological model:

If $\delta=-1$, from equation (25), we get

$$C_1\frac{S''}{S}-C_2\frac{S'^2}{S^2}-\frac{(4r-1)}{4}S^4-(r-1)S^{2n+2}-\frac{\omega}{2}k_1^{\,2}=0 \tag{39}$$

where $C_1=r(n-1)$,

$C_2=r(n-1)+n(n+2)$.

For $n=-1$, with a suitable substitution from equation (39), we get

$$S'^2=\gamma^2S^2-W^2S^6 \tag{40}$$

where $\gamma^2=\frac{2(r-1)+\omega k_1^{\,2}}{2}$, $r<0$ and $W^2=\frac{1}{4}$.

From equation (40), we get

$$S^2 = \left(\frac{\gamma}{W}\right)\text{sech}2\gamma T \tag{41}$$

From equations (22) and (41), we get

$$R^{-2} = \left(\frac{\gamma}{W}\right)\text{sech}2\gamma T \tag{42}$$

From equations (19), (26), (41) and (42), we get the energy density

$$\rho = -r\frac{\gamma}{W}\text{sech}2\gamma T\,,\;\; r<0 \tag{43}$$

From equations (17), (26), (41) and (42), we get the total pressure

$$\bar{p} = (1-r)\frac{\gamma}{W}\,\text{sech}\,2\gamma T \tag{44}$$

The proper pressure is given by

$$p = \epsilon_0\,\rho = -r\,\epsilon_0\,\frac{\gamma}{W}\,\text{sech}2\gamma T \tag{45}$$

From equations (18), (26), (41) and (44), we get the string tension density

$$\lambda = -\frac{\gamma}{W}\,\text{sech}2\gamma T \tag{46}$$

The particle energy density is given by

$$\rho_p = \rho - \lambda = (1-r)\frac{\gamma}{W}\,\text{sech}2\gamma T \tag{47}$$

The coefficient of the bulk viscosity is given by

$$\xi = \left[\frac{r(1-\epsilon_0)-1}{W}\right]\text{cosech}\,2\gamma T \tag{48}$$

The generalized mean Hubble parameter (H) is

$$H = \frac{1}{3}\left(H_1+H_2+H_3\right) = \frac{\gamma}{3}\tanh 2\gamma T \tag{49}$$

The metric (2.1), in this case, can be written as

$$\begin{aligned} ds^2 &= \left[\left(\frac{W}{\gamma}\right)\cosh 2\gamma T\right]^{\frac{1}{2}} dT^2 \\ &-\left[\left(\frac{W}{\gamma}\right)\cosh 2\gamma T\right](d\theta^2+\cosh^2\theta\, d\phi^2) \\ &-\left[\left(\frac{\gamma}{W}\right)\text{sech}2\gamma T\right](d\psi+\sinh\theta\, d\phi)^2 \end{aligned} \tag{50}$$

Thus, (50), together with (26), (43), (44), (46) and (48), constitutes a Bianchi type-VIII string cosmological model with bulk viscosity in Saez-Ballester (1986) theory of gravitation.

Bianchi type-IX ($\delta = 1$) cosmological model:

If $\delta = 1$, from equation (25), we get

$$\begin{aligned} C_1\frac{S''}{S} - C_2\frac{S'^2}{S^2} - \frac{(4r-1)}{4}S^4 \\ +(r-1)S^{2n+2} - \frac{\omega}{2}k_1{}^2 = 0 \end{aligned} \tag{51}$$

where $C_1 = r(n-1)$, $C_2 = r(n-1)+n(n+2)$.

For $n=-1$, with a suitable substitution from equation (50), we get

$$S'^2 = \gamma^2 S^2 - W^2 S^6 \tag{52}$$

where $\gamma^2 = \frac{2(1-r)+\omega k_1{}^2}{2}$ and $W^2 = \frac{1}{4}$.

From equation (52), we get

$$S^2 = \left(\frac{\gamma}{W}\right)Sech2\gamma T \tag{53}$$

From equations (22) and (53), we get

$$R^{-2} = \left(\frac{\gamma}{W}\right)Sech2\gamma T \tag{54}$$

From equations (19), (26), (53) and (54), we get the energy density

$$\rho = r\frac{\gamma}{W}Sech2\gamma T \tag{55}$$

From equations (17), (26), (53) and (54), we get the total pressure

$$\bar{p} = (r-1)\frac{\gamma}{W}Sech\,2\gamma T \tag{56}$$

The proper pressure is given by

$$p = \epsilon_0\,\rho = r\,\epsilon_0\,\frac{\gamma}{W}Sech2\gamma T \tag{57}$$

From equations (18), (26), (53) and (56), we get the string tension density

$$\lambda = \frac{\gamma}{W}Sech2\gamma T \tag{58}$$

The particle energy density is given by

$$\rho_p = \rho - \lambda = (r-1)\frac{\gamma}{W}Sech2\gamma T \tag{59}$$

The coefficient of bulk viscosity is given by

$$\xi = \left[\frac{r(1-\epsilon_0)-1}{W}\right]Co\sec h\,2\gamma T \tag{60}$$

The generalized mean Hubble parameter (H) is

$$H = \frac{1}{3}\left(H_1+H_2+H_3\right) = \frac{\gamma}{3}\tanh 2\gamma T \tag{61}$$

The metric (2.1), in this case, can be written as

$$\begin{aligned} ds^2 &= \left[\left(\frac{W}{\gamma}\right)\cosh 2\gamma T\right]^{\frac{1}{2}} dT^2 \\ &-\left[\left(\frac{W}{\gamma}\right)\cosh 2\gamma T\right](d\theta^2+\sin^2\theta\, d\phi^2) \\ &-\left[\left(\frac{\gamma}{W}\right)\text{sech}2\gamma T\right](d\psi+\cos\theta\, d\phi)^2 \end{aligned} \tag{62}$$

Thus, (62), together with (26), (55), (56), (58) and (60), constitutes a Bianchi type-IX string cosmological model with bulk viscosity in Saez-Ballester (1986) theory of gravitation.

ISOTROPIC COSMOLOGICAL MODELS:

Bianchi type-II ($\delta = 0$) cosmological model:

If $\delta = 0$, from equation (25), we get

$$C_1\frac{S''}{S} - C_2\frac{S'^2}{S^2} - \frac{(4r-1)}{4}S^4 - \frac{\omega}{2}k_1{}^2 = 0 \tag{63}$$

where $C_1 = r(n-1)$, $C_2 = r(n-1) + n(n+2)$.

For $n=1$, with a suitable substitution from equation (63), we get

$$S'^2 = W^2 S^6 - \gamma^2 S^2 \tag{64}$$

where $W^2 = \frac{1-4r}{12}, r < 0$ and $\gamma^2 = \frac{\omega k_1^2}{6}$.

From equation (64), we get

$$S^2 = R^2 = \left(\frac{\gamma}{W}\right)\sec 2\gamma T \tag{65}$$

From equations (19), (26) and (65), we get the energy density

$$\rho = -r\frac{W}{\gamma}\cos 2\gamma T \tag{66}$$

From equations (17), (26) and (65), we get the total pressure

$$\bar{p} = \frac{(r-1)}{3}\frac{W}{\gamma}\cos 2\gamma T \tag{67}$$

The proper pressure is given by

$$p = \epsilon_0 \rho = -r \epsilon_0 \frac{W}{\gamma}\cos 2\gamma T \tag{68}$$

From equations (18), (26), (65) and (67), we get the string tension density

$$\lambda = -\frac{W}{\gamma}\cos 2\gamma T \tag{69}$$

The particle energy density is given by

$$\rho_p = \rho - \lambda = (1-r)\frac{W}{\gamma}\cos 2\gamma T \tag{70}$$

The coefficient of bulk viscosity is given by

$$\xi = \left[\frac{1 - r(1 + 3\epsilon_0)}{9}\right]\frac{W}{\gamma^2}\cos 2\gamma T \cot 2\gamma T \tag{71}$$

The generalized mean Hubble parameter (H) is

$$H = \frac{1}{3}\left(H_1 + H_2 + H_3\right) = \gamma \tan 2\gamma T \tag{72}$$

The metric (2.1), in this case, can be written as

$$ds^2 = \left[\left(\frac{\gamma}{W}\right)\sec 2\gamma T\right]^{\frac{3}{2}} dT^2 - \left[\left(\frac{\gamma}{W}\right)\sec 2\gamma T\right] \left[(d\theta^2 + d\phi^2) + (d\psi + \theta\, d\phi)^2\right] \tag{73}$$

Thus, (73) together with (26), (66), (67), (69) and (71) constitutes a Bianchi type-II string cosmological model with bulk viscosity in the isotropic form in Saez-Ballester [17] theory of gravitation.

Bianchi type-VIII ($\delta = 0$) cosmological model:

If $\delta = -1$, from equation (25), we get

$$C_1\frac{S''}{S} - C_2\frac{S'^2}{S^2} - \frac{(4r-1)}{4}S^4 - (r-1)S^{2n+2} - \frac{\omega}{2}k_1^2 = 0 \tag{74}$$

where $C_1 = r(n-1)$, $C_2 = r(n-1) + n(n+2)$.

For $n=1$, with a suitable substitution, from equation (74), we get

$$S'^2 = W^2 S^6 - \gamma^2 S^2 \tag{75}$$

where $W^2 = \frac{5-8r}{12}, r < 0$ and $\gamma^2 = \frac{\omega k_1^2}{6}$.

From equation (75), we get

$$S^2 = R^2 = \left(\frac{\gamma}{W}\right)\sec 2\gamma T \tag{76}$$

From equations (19), (26) and (76), we get the energy density

$$\rho = -2r\frac{W}{\gamma}\cos 2\gamma T \tag{77}$$

From equations (3.6), (26) and (76), we get the total pressure

$$\bar{p} = 2\frac{(r-1)}{3}\frac{W}{\gamma}\cos 2\gamma T \tag{78}$$

The proper pressure is given by

$$p = \epsilon_0 \rho = -2r \epsilon_0 \frac{W}{\gamma}\cos 2\gamma T \tag{79}$$

From equations (18), (26), (76) and (78), we get the string tension density

$$\lambda = -2\frac{W}{\gamma}\cos 2\gamma T \tag{80}$$

The particle energy density is given by

$$\rho_p = \rho - \lambda = 2(1-r)\frac{W}{\gamma}\cos 2\gamma T \tag{81}$$

The coefficient of the bulk viscosity is given by

$$\xi = \left[\frac{-2[1 + r(3\epsilon_0 - 1)]}{9}\right]\frac{W}{\gamma^2}\cos 2\gamma T \cot 2\gamma T \tag{82}$$

The generalized mean Hubble parameter (H) is

$$H = \frac{1}{3}\left(H_1 + H_2 + H_3\right) = \gamma \tan 2\gamma T \tag{83}$$

The metric (2.1), in this case, can be written as

$$ds^2 = \left[\left(\frac{\gamma}{W}\right)\sec 2\gamma T\right]^{\frac{3}{2}} dT^2 - \left[\left(\frac{\gamma}{W}\right)\sec 2\gamma T\right] \left[(d\theta^2 + \cosh^2\theta\, d\phi^2) + (d\psi + \sinh\theta\, d\phi)^2\right] \tag{84}$$

Thus, (84) together with (26), (77), (78), (80) and (82) constitutes a Bianchi type-VIII string cosmological model with bulk viscosity in the isotropic form in Saez-Ballester [17] theory of gravitation.

Bianchi type-IX ($\delta = 1$) cosmological model:

If $\delta = 1$, from equation (25), we get

$$C_1\frac{S''}{S} - C_2\frac{S'^2}{S^2} - \frac{(4r-1)}{4}S^4 + (r-1)S^{2n+2} - \frac{\omega}{2}k_1^2 = 0 \tag{85}$$

where $C_1 = r(n-1)$, $C_2 = r(n-1) + n(n+2)$.

For $n=1$, with a suitable substitution, from equation (85), we get

$$S'^2 = \gamma^2 S^2 - W^2 S^6 \tag{86}$$

where $W^2 = \frac{1}{4}$ and $\gamma^2 = \frac{-\omega k_1^2}{6}, \omega < 0$.

From equation (86), we get

$$S^2 = R^2 = \left(\frac{\gamma}{W}\right)\text{sech}\, 2\gamma T \tag{87}$$

From equations (17) to (19), (26) and (87), we can observe that the energy density, ρ, the total pressure, $\bar{p}$, the string tension density, λ, and the coefficient of bulk viscosity, ξ, will vanish.

The metric (7), in this case, can be written as

$$ds^2 = \left[\left(\frac{\gamma}{W}\right)\sec 2\gamma T\right]^{\frac{3}{2}} dT^2 - \left[\left(\frac{\gamma}{W}\right)\sec 2\gamma T\right] \tag{88}$$

$$[(d\theta^2 + \sin^2\theta\, d\phi^2) + (d\psi + \cos\theta\, d\phi)^2]$$

Thus, (88), together with (26) and (87), constitutes a Bianchi type-IX vacuum cosmological model in the isotropic form in Saez-Ballester [17] theory of gravitation.

4. Some other important properties of the Model:

Bianchi type-II cosmological model ($\delta = 0$):

- The spatial volume for the model is

$$V = (-g)^{\frac{1}{2}} = \left[\left(\frac{\gamma}{W}\right)\sec 2\gamma T\right]^{\frac{2n+1}{2}} \tag{89}$$

- The average scale factor for the model is

$$a = V^{\frac{1}{3}} = \left[\left(\frac{\gamma}{W}\right)\sec 2\gamma T\right]^{\frac{2n+1}{6}} \tag{90}$$

- The expression for the expansion scalar θ is calculated for the flow vector u^i and given by

$$\theta = u^i{}_{;i} = -(2n+1)\gamma \tan 2\gamma T \tag{91}$$

and the shear scalar σ is given by

$$\sigma^2 = \frac{1}{2}\sigma^{ij}s_{ij} = \frac{(2n+1)^2}{2}\gamma^2 \tan^2 2\gamma T \tag{92}$$

- **Deceleration Parameter:**

The deceleration parameter q is given by

$$q = (-3\theta^{-2})(\theta_{,i}u^i + \frac{1}{3}\theta^2) = -\left(\frac{6}{2n+1}\right)\text{co}\sec^2 2\gamma T - 1 \tag{93}$$

From equation (93), we can observe that the deceleration parameter q is

always negative for $n > 0$ and hence, it represents the accelerating universe.

- The average anisotropy parameter is defined by

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\Delta H_i}{H}\right)^2 = \frac{2(n-1)^2}{(2n+1)^2} \tag{94}$$

where $\Delta H_i = H_i - H \quad (i = 1,2,3)$.

- **Jerk Parameter:**

Jerk parameter in cosmology is defined as the dimensionless third derivative of scale factor with respect to cosmic time; it is given by

$$j(t) = \frac{1}{H^3}\frac{\ddot{a}}{a} \tag{95}$$

We can rewrite equation (94) as

$$j = q + 2q^2 - \frac{\dot{q}}{H} \tag{96}$$

where q is the deceleration parameter.

Hence, the expression for the jerk parameter is given by

$$j(t) = \frac{9(4n+10)}{(2n+1)^2}\text{co}\sec^2 2\gamma T + 1 \tag{97}$$

- **Age of the universe:**

The look-back time, $\Delta t = t_0 - t(z)$, is the difference between the age of the universe at the present time (z=0) and the age of the universe when a particular light ray is at the redshift z, and the expansion scalar of the universe $a(t_z)$ is related to a_0 by $1 + z = \frac{a_0}{a}$, where a_0 is the present scale factor. Therefore, from (90), we get

$$1 + z = \frac{a_0}{a} = \left(\frac{\gamma/_w \sec 2\gamma T_0}{\gamma/_w \sec 2\gamma T}\right)^{2n+1/6} \tag{98}$$

The above equation gives

$$\cos 2\gamma T = (1+z)^{6/2n+1}\cos 2\gamma T_0 \tag{99}$$

Taking limit $z \to \infty$ in (99), we get the age of the universe as

$$T_0 = \frac{\pi}{4k_1}\sqrt{\frac{2n(n+2)}{\omega}} \tag{100}$$

This is different from the present estimate, i.e. $T_0 = 14\,Gyr$. But if we take different values of $n \,\&\, \omega$ with $k_1 = 1$, then the derived model is in a good agreement with the present age of the universe, as shown in figure 1.

Bianchi type-VIII ($\delta = -1$) and IX ($\delta = 1$) cosmological models:

- The spatial volume for both models (50) and (62) is

$$V = \left[\left(\frac{\gamma}{W}\right)\text{sech} 2\gamma T\right]^{\frac{-1}{2}} f(\theta), \tag{101}$$

where

$\gamma^2 = \frac{2(r-1)+\omega k_1^2}{2} \,\&\, W^2 = \frac{1}{4}$, $\gamma^2 = \frac{2(1-r)+\omega k_1^2}{2} \,\&\, W^2 = \frac{1}{4}$, $f(\theta) = \cosh\theta \;\&\; \sin\theta$ for Bianchi type-VIII and IX, respectively.

- The average scale factor for both models (50) and (62) is

$$a = V^{1/3} = \left[\left(\frac{\gamma}{W}\right)\text{sech} 2\gamma T\right]^{\frac{-1}{6}} [f(\theta)]^{1/3} \tag{102}$$

- The expression for the expansion scalar θ and the shear scalar σ for the models (50) and (62) are given by

$$\theta = \gamma\, \text{Tanh} 2\gamma T \tag{103}$$

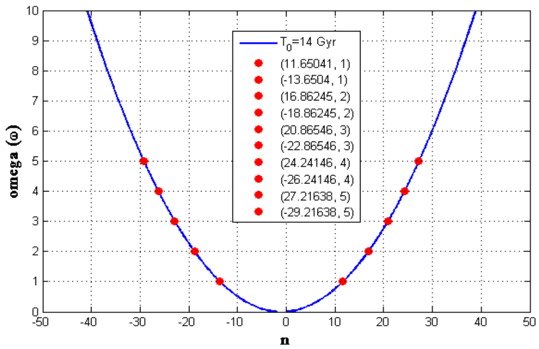

Figure 1. (co Plot of the age of the universe.

$$\sigma^2 = \frac{\gamma^2}{2}\mathrm{Tanh}^2 2\gamma T \qquad (104)$$

- **Deceleration Parameter:**

The deceleration parameter q for the models (50) and (62) is given by

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = -6\cos\mathrm{ech}^2 2\gamma T - 1 \qquad (105)$$

From equation (105), we can observe that the deceleration parameter q is negative for $T \neq 0$ and hence, they represent the accelerating universe.

- The average anisotropy parameter for the models (50) and (62) is defined by

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\Delta H_i}{H}\right)^2 = 8, \text{ where}$$

$$\Delta H_i = H_i - H \quad (i = 1,2,3) \qquad (106)$$

- **Jerk Parameter:**

The expression for the jerk parameter is given by

$$j(t) = -54\cos\mathrm{ech}^2 2\gamma T - 1 \qquad (107)$$

- **Age of the universe:** The look-back time, $\Delta t = t_0 - t(z)$, is the difference between the age of the universe at the present time (z=0) and the age of the universe when a particular light ray is at the redshift z, and the expansion scalar of the universe $a(t_z)$ is related to a_0 by $1+z = \frac{a_0}{a}$, where a_0 is the present scale factor. Therefore, from (102) ,we get

$$1+z = \frac{a_0}{a} = \left(\frac{\gamma/_w \,\mathrm{sech}2\gamma T_0}{\gamma/_w \,\mathrm{sech}2\gamma T}\right)^{-1/6} \qquad (108)$$

The above equation gives

$$\cosh 2\gamma T = \cosh 2\gamma T_0 (1+z)^{-6} \qquad (109)$$

Taking limit $z \to \infty$ in (109), we get

$$T_0 = \frac{\pi}{4}\sqrt{\frac{2}{2(r-1)+\omega k_1^2}} \qquad (110)$$

By giving suitable values to $r, k_1 \,\&\, \omega$ in (110), we can always get the present age of the universe.

Bianchi type- II and VIII Isotropic cosmological models:

- The spatial volume for the models is

$$V = \left[\left(\frac{\gamma}{W}\right)\sec 2\gamma T\right]^{\frac{3}{2}} f(\theta) \qquad (111)$$

where $\gamma^2 = \frac{\omega k_1^2}{6}, W^2 = \frac{1-4r}{12} \,\&\, f(\theta) = 1$ for Bianchi type- II metric,

$\gamma^2 = \frac{\omega k_1^2}{6}, W^2 = \frac{5-8r}{12} \,\&\, f(\theta) = \cosh\theta$ for Bianchi type- VIII metric.

- The average scale factor for the models is

$$a = V^{1/3} = \left[\left(\frac{\gamma}{W}\right)\sec 2\gamma T\right]^{\frac{1}{2}} [f(\theta)]^{1/3} \qquad (112)$$

- The expression for expansion scalar θ and the shear scalar σ for the models are given by

$$\theta = 3\gamma \tan 2\gamma T \qquad (113)$$

$$\sigma^2 = \frac{9\gamma^2}{2}\tan^2 2\gamma T \qquad (114)$$

- **Deceleration Parameter:**

The deceleration parameter q for the models is given by

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = -2\cos\mathrm{ec}^2 2\gamma T - 1 \qquad (115)$$

From equation (115), we can observe that the deceleration parameter q is negative for $T \neq 0$ and hence, they represent the accelerating universe.

- The average anisotropy parameter for the models are defined by

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\Delta H_i}{H}\right)^2 = 0, \text{ where}$$

$$\Delta H_i = H_i - H \quad (i = 1,2,3) \qquad (116)$$

- **Jerk Parameter:**

The expression for the jerk parameter is given by

$$j(t) = 14\cos\mathrm{ec}^2 2\gamma T + 1 \qquad (117)$$

- **Age of the universe:**

The look-back time, $\Delta t = t_0 - t(z)$ is the difference between the age of the universe at the present time (z=0) and the age of the universe when a particular light ray is at the red shift z, and the expansion scalar of the universe $a(t_z)$ is related to a_0 by $1+z = \frac{a_0}{a}$, where a_0 is the present scale factor. Therefore, from (112) ,we get

$$1+z = \frac{a_0}{a} = \left(\frac{\gamma/_w \,\sec 2\gamma T_0}{\gamma/_w \,\sec 2\gamma T}\right)^{1/2} \qquad (118)$$

The followings are the observations and conclusions:

	Anisotropic universe		Isotropic universe
	Bianchi type-II metric	Bianchi type-VIII andIX metrics	Bianchi type –II and VIII metrics
Singularity	Initial singularity at T=0 for $n>0$	No singularities	Singularities at $T=\frac{(2r+1)\pi}{4\gamma}$, $r=0, \pm 1, \pm 2, \ldots$
Spatial volume V	Varies with time T and θ	Varies with time T and θ	Varies with time T and θ
Expansion scalar θ	Increases with the rise of time	Increases with the rise of time	Increases with the rise of time
Shear scalar σ	Increases with the rise of time	Increases with the rise of time	Increases with the rise of time
Hubble parameter H	Increases with the rise of time	Increases with the rise of time	Increases with the rise of time
Energy density ρ , Total pressure $\overline{p}$, String density λ and Coefficient of bulk viscosity ξ	Tends to unity as $T\to 0$ and zero as $T\to\infty$	Tends to unity as $T\to 0$ and zero as $T\to\infty$	Tends to unity as $T\to 0$ and zero as $T\to\infty$
Deceleration parameter q	$q<0$ Accelerating	$q<0$ Accelerating	$q<0$ Accelerating

The above equation gives

$$\cos 2\gamma T = \cos 2\gamma T_0 (1+z)^2 \tag{119}$$

Taking limit $z\to\infty$ in (119), we get

$$T_0 = \frac{\pi}{4}\sqrt{\frac{6}{\omega k_1^2}} \tag{120}$$

By giving suitable values to k_1 & ω in (120), we can always get the present age of the universe.

- The vorticity tensor $w_{ij} = u_{i,j} - u_{j,i}$ is a measure of the rotation of the local rest-frame relative to the compass of inertia, which is identically zero. Hence, the fluid filling the universes are non-rotational.

5. Conclusions:

In this paper, we have presented homogeneous Bianchi type-II, VIII and IX anisotropic as well as isotropic bulk viscous string cosmological models in a theory of gravitation proposed by Saez and Ballester [17].

In standard cosmology, data tells us that the universe is homogeneous and isotropic. However, after the discovery of temperature anisotropies of the Cosmic Microwave background [33] radiation, it is conjectured that there are unobservable small amounts of anisotropies present in the early stages of the evolution of the universe. The presence of this feature seems to be inconsistent with the isotropic FRW model. The current CMB data supports an inflationary big bang model of cosmic origin for our universe. Hence, in our model, there is an unobservable small amount of anisotropy. There are corresponding studies regarding the calculation of bounds on the cosmological parameters. Of course, this will lead to the discussion of dark energy via viscosity. For this one, can refer to the article by Bamba *et al*. (2012).

It is well known that scalar field and bulk viscosity serve a significant role in getting an accelerated universe. The anisotropic and isotropic exact models presented here are new and more general, representing not only the early stage of evolution, but also the present universe.

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