Improving the Upper Bound on the Scaling Dimension in 2 Dimensional CFT

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Abstract. Modular invarinat, constraints the spectrum of the theory. Using the medum temprature expansion, for first and third order of derivative, a universal upper bound on the lowest primary field has been obtained in recent researches. In this paper, we will improve the upper bound on the scaling dimension of the lowest primary field. We use by the medium temprature expansion for an arbitrary order of derivatives. We show that the upper bound depends on the order of derivative. In this research, we obtain the optimal values of the order of derivatives which leads to the best upper bound.

Keywords: Conformal Field Theory, Modular Invariant, Primary Fields, Medium Temprature Expansion.

1. Introduction

One of the important issues in conformal field theory is fixing the theory without relying on the Lagrangian. This is the subject of Bootstrap project[1,2,3]. Using the constraints and symmetries which are imposed on the theory, the universal feature will be revealed.

Crossing symmetry is one of the constraints imposed on the conformal field theory. In four dimensional CFT by decomposition of the four-point function into the conformal block [4,5], and using the crossing symmetry, an upper bound on the weights of the fields that appear in the operator product expansion of two scalar operators has been obtained [6-10]. Similarly, a lower bound on the stress tensor central charge has been obtained [11, 12].

In two-dimension, beside the crossing symmetry, the modular invariance is a powerful constraint that helps us to know much more about the density of states and the spectrum of the theory. The disconnected diffeomorphism group of the torus is a modular group $\mathbf{PSL}(2,\mathbb{Z})$:

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{PSL}(2,\mathbb{Z}). \tag{1}$$

where $\tau = \tau_1 + i\tau_2$ is the complex structure which lies in the upper half plane($\tau_2 > 0$), and $\overline{\tau} = \tau_1 - i\tau_2$. The generators of the modular group are $T := (\tau,\overline{\tau}) \to (\tau+1,\overline{\tau}+1)$, and $S := (\tau,\overline{\tau}) \to \left(-\frac{1}{\tau},-\frac{1}{\overline{\tau}}\right)$:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

Since, the partition function depends only on the conformal structure τ and $\overline{\tau}$, therefore, it must be invariant under modular transformations. From invariance of partition function under T transformation, we can see that the differences between left and right conformal dimension is integer and the difference between the left and right central charges is multiple of 24 . The invariance under S transformation is a more powerful constraint and leads to the set of constraints on the density of states [13-15] and the spectrum of the theory[16-25].

Recently, using the modular invariance of partition function, an upper bound on the scaling dimension of primary fields has been obtained. For holomorphically factorizable partition function with $c_L, c_R \in 24\mathbb{Z}$, the holomorphic and antiholomorphic partition function is modular invariant. In this class of CFT, the lowest primary field is left moving or right moving and bounded from above as follows:

$$\Delta \leq \min\left(\frac{c_L}{24}+1, \frac{c_R}{24}+1\right). \tag{3}$$

In [16], considering S invariance of partition function at the self-dual point $\tau = -\frac{1}{\tau} = i$, an interesting set of constraints on the partition function has been obtained. In [16] by considering the neighborhood of $\tau = -\overline{\tau} = i$

$$\tau \equiv ie^{s}, \qquad \overline{\tau} \equiv -ie^{-s}. \tag{4}$$

and taking the derivatives of the S invariance constraint of partition function at $s = 0$:

$$\left(\frac{\partial}{\partial s}\right)^{N_R}\left(\frac{\partial}{\partial \overline{s}}\right)^{N_L} Z(\mathrm{ie}^{s}, -\mathrm{ie}^{\overline{s}})\Big|_{s=0} = \left(\frac{\partial}{\partial s}\right)^{N_R}\left(\frac{\partial}{\partial \overline{s}}\right)^{N_L} Z(\mathrm{ie}^{-s}, -\mathrm{ie}^{-\overline{s}})\Big|_{s=0}$$

a set of constraints on the partition function obtained as follows

$$\left(\tau\frac{\partial}{\partial \tau}\right)^{N_R}\left(\overline{\tau}\frac{\partial}{\partial \overline{\tau}}\right)^{N_L} Z(\tau, \overline{\tau})\Big|_{\tau=-\overline{\tau}=i} = 0 \qquad \text{for} \quad N_L + N_R = \text{odd}. \tag{5}$$

which is called *medium temperature expansion*.

e Using th medium temperature expansion in CFT with $c_L, c_R > 1$, an upper bound on the lowest primary field obtained as follows[24]

$$\Delta \leq \frac{c_{\text{tot}}}{12} + 0.4755, \qquad c_{\text{tot}} := c_L + c_R. \tag{6}$$

It was shown for any primary fields with conformal dimension Δ_n with $n \leq e^{\frac{\pi c_{tot}}{12}}$, the similar bound has been obtained in the large central charge limit[22]. The bound in the holomorphically factorizable case (5), is a factor of two lower than the bound in the general case (6). Similar to the holomorphically factrorizable case, in special class of (2,2) supersymmetric theories in the large central charge limit, the similar bound has been obtained [26]. Therefore, one suspects that the bound (6) can be improved. Using the medium temperature expansion method and ST invariance of partition function, it was shown that the upper bound on the primary fields with even spins has been improved by a factor of 2 [23].

The linear functional method in the large central charge limit, is used in order to improve the bound (6) as follows

$$\Delta \leq \frac{c_{\text{tot}}}{12} - \frac{1}{6} - \frac{1}{2\pi} + \frac{2}{e^{2\pi}-1}. \tag{7}$$

In obtaining the bounds (6) and (7) the first and third order derivatives of partition function has been applied in the canonical ensemble.

In this work in order to improve the upper bound, we use the medium temperature expansion in the grand canonical ensemble for an arbitrary value of N_L, N_R. Using this constraint, we show that by increasing the order of derivative better upper bound is obtained. However, the order of derivative cannot be increased arbitrarily. There are some constraints about derivatives. We obtain the optimal values of the derivatives, which lead to the better upper bound:

$$\Delta_1 \leq \frac{c_{tot}}{12} - \frac{(N_L'^2 + N_R'^2 - N_L^2 - N_R^2)}{\pi(N_L + N_R - N_L' - N_R')} - \frac{1}{6} - \frac{1}{2\pi} + \frac{2}{e^{2\pi} - 1}. \tag{8}$$

This paper is organized as follows. In section 2, we investigate the partition function of the CFTs with $c_{L,} c_R > 1$. In section 3, we use the medium temperature expansion for an arbitrary value of N_L, N_R in the grand canonical ensemble. We obtain the optimal values of the derivatives, which leads to a better upper bound. In section 4, we reviewing our results and draw our conclusions. In appendices, we have collected the detail of some computations.

2. Decomposition of Partition function

The partition function of a general unitary two-dimensional CFT on a torus with complex structure $\tau = \tau_1 + i\tau_2$, can be written as follows

$$Z(\tau,\bar{\tau}) = Tr(e^{2i\pi\tau(L_0 - \frac{c_L}{24})} e^{-2i\pi\bar{\tau}(\bar{L}_0 - \frac{c_R}{24})}) = \sum_{h,\bar{h}=0} \rho(h,\bar{h}) e^{2i\pi\tau(h - \frac{c_L}{24})} e^{-2i\pi\bar{\tau}(\bar{h} - \frac{c_R}{24})}, \tag{9}$$

where c_L and c_R are left and right central charges respectively. L_0 and $\bar{L}_0$ are zeroth left-moving and right-moving operators with eigenvalues h and $\bar{h}$ which satisfy Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c_L}{12}(m^3 - m)\delta_{m,-n}, \tag{10}$$

$$[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \frac{c_R}{12}(m^3 - m)\delta_{m,-n}. \tag{11}$$

(11)

The Hilbert space of state of CFT_2 is characterized by the weight of the primary fields of the theory. The effect of L_n with $n < 0$ on the highest weight state (which corresponds to primary operator) create descendants of a primary field. For $c_L > 1$ and $h \neq 0$, for any set of n_i with $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$, these states are linearly independent:

$$L_{-n_1} L_{-n_2} \cdots L_{-n_k} |0\rangle. \tag{12}$$

For $c_L > 1$, and $h = 0$ the linearly independent states are given by

$$L_{-n_1} L_{-n_2} \cdots L_{-n_k} |0\rangle. \tag{13}$$

where $n_1 \geq n_2 \geq \cdots \geq n_k \geq 2$. Therefore, in the partition function we separate the contribution of identity from other primary operators

$$Z(\tau,\bar{\tau}) = Z^{id}(\tau,\bar{\tau}) + \sum_A Z^A(\tau,\bar{\tau}), \tag{14}$$

Where sum is over all the primary operators except the identity operator. For CFT_2 with $c_L, c_R > 1$ the partition function can be written in terms of Virasoro character

$$Z^{(id)}(\tau,\bar{\tau}) = \chi_0(\tau)\bar{\chi}_0(\bar{\tau}). \tag{15}$$

$$Z^A(\tau,\bar{\tau}) = \chi_{h_A}(\tau)\bar{\chi}_{\bar{h}_A}(\bar{\tau}). \tag{16}$$

with

$$\chi_h(\tau)\bar{\chi}_{\bar{h}}(\bar{\tau}) = \begin{cases} \frac{(1-\bar{q})(1-q)}{|\eta(\tau)|^2}\bar{q}^{-\bar{E}_0}q^{-E_0}, & \bar{h}=0, h=0, \\ \frac{(1-q)}{|\eta(\tau)|^2}\bar{q}^{\bar{h}-\bar{E}_0}q^{h-E_0} & \bar{h}>0, h=0, \\ \frac{(1-\bar{q})}{|\eta(\tau)|^2}\bar{q}^{\bar{h}-\bar{E}_0}q^{h-E_0} & \bar{h}=0, h>0, \\ \frac{1}{|\eta(\tau)|^2}\bar{q}^{\bar{h}-\bar{E}_0}q^{h-E_0} & \bar{h}>0, h>0. \end{cases} \tag{17}$$

Therefore, we can write the characters as follows

$$\chi_h(\tau) = \frac{q^{h+E_0}}{\eta(\tau)}(1-\mathrm{q})^{\delta_{h,0}}, \qquad \chi_0(\tau) = (1-q)\frac{q^{E_0}}{\eta(\tau)}, \tag{18}$$

where $q = e^{2\pi i\tau}$, $E_0 = \frac{1-c_L}{24}$, $\bar{E}_0 = \frac{1-c_R}{24}$ and $\eta(\tau)$ is Dedekind eta function.

3. Universal upper bound on the lowest primary field

Using the decomposition of partition function (14), we rewrite the medium temperature expansion(5), in the useful configuration as follows

$$\left.\frac{\left(\tau\frac{\partial}{\partial\tau}\right)^{N_L}\left(\bar{\tau}\frac{\partial}{\partial\bar{\tau}}\right)^{N_R}\sum_A Z^A(\tau,\bar{\tau})}{\left(\tau\frac{\partial}{\partial\tau}\right)^{N'_L}\left(\bar{\tau}\frac{\partial}{\partial\bar{\tau}}\right)^{N'_R}\sum_A Z^A(\tau,\bar{\tau})}\right|_{\tau=+i} = \left.\frac{\left(\tau\frac{\partial}{\partial\tau}\right)^{N_L}\left(\bar{\tau}\frac{\partial}{\partial\bar{\tau}}\right)^{N_R} Z^{(id)}(\tau,\bar{\tau})}{\left(\tau\frac{\partial}{\partial\tau}\right)^{N'_L}\left(\bar{\tau}\frac{\partial}{\partial\bar{\tau}}\right)^{N'_R} Z^{(id)}(\tau,\bar{\tau})}\right|_{\tau=+i}. \tag{19}$$

In order to use this constraint, first let us take the derivatives of $Z^A(\tau,\bar{\tau})$

$$\left(\tau\frac{\partial}{\partial\tau}\right)^N Z^A(\tau,\bar{\tau})\,|_{\tau=i} = (-1)^N g^{(N)}(h_A + \mathrm{E}_0)\chi_{h_A}(i)\bar{\chi}_{\bar{h}_A}(-i), \tag{20}$$

where $g^{(N)}(h)$ is a polynomial which is defined as follows

$$\left.\left(\tau\frac{\partial}{\partial\tau}\right)^N \chi_h(\tau)\right|_{\tau=i} = (-1)^N g^{(N)}(h+E_0)\chi_h(i). \tag{21}$$

In what follows, it will be convenient to write the polynomial $g^{(N)}(h)$ as follows

$$g^{(N)}(h) = \sum_{n=0}^{N}(-1)^n A_{N-n}^{(N)}(i)\left(2\pi h - \frac{1}{4} - \frac{2\pi\delta_{h,0}}{e^{2\pi}-1}\right)^{N-n}, \tag{22}$$

where $A_N^{(N)}(i) = 1$, and the other $A_N^{(N)}(i)$ s can be obtained from the recursion formula as follows

$$A_n^{(N)}(\tau) = \tau\frac{\partial A_n^{(N-1)}}{\partial\tau} - (n+1)\left(\tau^2\frac{\partial}{\partial\tau}\frac{\eta'(\tau)}{\eta(\tau)} + \frac{(2\pi i\tau)^2 e^{-2\pi i\tau}}{e^{-2\pi i\tau}-1}\right)A_{n+1}^{(N-1)} + nA_n^{(N-1)} + A_{n-1}^{(N-1)}. \tag{23}$$

The details of calculation are mentioned in Appendix A.

It is easy to solve these recursion formula. $A_{N-1}^{(N)}(\tau)$ and $A_{N-2}^{(N)}(\tau)$ are given by

$$
\begin{aligned}
&A_{N-1}^{(N)}(\tau)=\frac{N(N-1)}{2},\\
&A_{N-2}^{(N)}(\tau)=\frac{(N-1)^2(N-2)^2}{2\times 4}+\frac{(N-1)(N-2)(2N-3)}{2\times 6}-\frac{N(N-1)}{2}\left(\tau^2\frac{\partial}{\partial\tau}\frac{\eta'(\tau)}{\eta(\tau)}+\frac{(2\pi i\tau)^2 e^{-2\pi i\tau}}{e^{-2\pi i\tau}-1}\right).
\end{aligned} \tag{24}
$$

Using (15) and (20) one can take the derivative of $Z^{(id)}(\tau,\bar{\tau})$:

$$
\left(\tau\frac{\partial}{\partial\tau}\right)^N Z^{id}(\tau,\bar{\tau})\Big|_{\tau=i}=(-1)^N g^{(N)}(E_0)\chi_0(i)\overline{\chi_0(-i)}. \tag{25}
$$

Inserting (20) and (25) into (19) yields

$$
\frac{\sum_{A=1}^{\infty} g^{(N_L)}(h_A+E_0)g^{(N_R)}(\bar{h}_A+\bar{E}_0)\Lambda_A e^{-2\pi\Delta_A}}{\sum_{B=1}^{\infty} g^{(N'_L)}(h_B+E_0)g^{(N'_R)}(\bar{h}_B+\bar{E}_0)\Lambda_B e^{-2\pi\Delta_B}}=\frac{g^{(N_L)}(E_0)g^{(N_R)}(\bar{E}_0)}{g^{(N'_L)}(E_0)g^{(N'_R)}(\bar{E}_0)}:=G_0(E_0,\bar{E}_0), \tag{26}
$$

where,

$$
\Lambda_A=(1-e^{2\pi})^{\delta_{h_A,0}+\delta_{\bar{h}_A,0}}. \tag{27}
$$

Subtracting both sides of (26) results to

$$
\frac{\sum\left[g^{(N_L)}(h_A+E_0)g^{(N_R)}(\bar{h}_A+\bar{E}_0)-G_0(E_0,\bar{E}_0)g^{(N'_L)}(h_A+E_0)g^{(N'_R)}(\bar{h}_A+\bar{E}_0)\right]\Lambda_A e^{-2\pi\Delta_A}}{\sum_{B=1}^{\infty} g^{(N'_L)}(h_B+E_0)g^{(N'_R)}(\bar{h}_B+\bar{E}_0)\Lambda_B e^{-2\pi\Delta_B}}=0 \tag{28}
$$

In the following, we assume $c_L=c_R:=c$. In order to proceed, we write the conformal weight h and $\bar{h}$ in terms of the scaling dimension Δ, and the spin j :

$$
\Delta=h+\bar{h},\qquad j=h-\bar{h}. \tag{29}
$$

Therefore, (28) can be written as follows

$$
\frac{\sum_{A=1}^{\infty} G(\Delta_A,j_A)\Lambda_A e^{-2\pi\Delta_A}}{g^{N_L}(E_0)g^{N'_R}(E_0)\sum_B g^{N'_L}\left(\frac{\Delta+j}{2}+E_0\right)g^{N'_R}\left(\frac{\Delta-j}{2}+E_0\right)\Lambda_B e^{-2\pi\Delta_B}}=0, \tag{30}
$$

where,

$$
\begin{aligned}
G(\Delta,\mathrm{j})=\;&g^{(N_L)}((\Delta+j)/2+E_0)g^{(N_R)}((\Delta-j)/2+E_0)g^{(N'_L)}(E_0)g^{(N'_R)}(E_0)\\
&-(N_L\leftrightarrow N'_L,N_R\leftrightarrow N'_R).
\end{aligned} \tag{31}
$$

Since $G(\Delta,\mathrm{j})$ is an odd function of Δ, it has at least one real root. $G(\Delta,\mathrm{j})$ is a function of Δ and j. In unitary CFT, from $h\geq o$ and $\bar{h}\geq o$, we conclude $-\Delta\leq j\leq\Delta$. Therefore, the roots of $G(\Delta,\mathrm{j})$ depends on the value of spin. Let us denote the largest real root of $G(\Delta,\mathrm{j})$ for $j\approx\mathcal{O}(1)$ by Δ^+ and for $j\approx\mathcal{O}(\Delta)$ by $\bar{\Delta}^+$. In the limit of $\Delta\to\infty$, $G(\Delta,\mathrm{j})$ goes to infinity, therefore, for $\Delta>\max(\Delta^+,\bar{\Delta}^+)$, the function $G(\Delta,\mathrm{j})$ is positive. We show in the Appendix B, $\max(\Delta^+,\bar{\Delta}^+)=\Delta^+$. Therefore, for $\Delta>\Delta^+$, $G(\Delta,\mathrm{j})$ is positive.

Now, suppose that in (30):

$$
0=\Delta_0<\Delta_1\leq\Delta_2\leq\Delta_3\leq\cdots
$$

Hence, from

$$
\Delta_n\geq\Delta_1>\Delta^+,\qquad \text{all}\,\mathrm{n}>0. \tag{32}
$$

we verify that

$$G(\Delta_n, \mathrm{j}_n) > 0 \qquad \text{all } n > 0. \tag{33}$$

Now, suppose that $\tilde{\Delta}^+$ is the largest real root of $g^{N_L'}\left(\frac{\Delta+j}{2}+E_0\right)g^{N_R'}\left(\frac{\Delta-j}{2}+E_0\right)=0$. Similarly, for

$$\Delta_n \geq \Delta_1 > \tilde{\Delta}^+, \qquad \text{all n} > 0. \tag{34}$$

We have

$$g^{N'_L}\left(\frac{\Delta_n+j_n}{2}+E_0\right)g^{N_R'}\left(\frac{\Delta_n-j_n}{2}+E_0\right) > 0 \qquad \text{all } n > 0. \tag{35}$$

Consequently, for $\Delta_1 > \max(\Delta^+, \tilde{\Delta}^+)$, every terms in the numerator and denominator of the right hand side of (30) are positive. It is in contrast with the left hand side of (30). Thus, the hypothesis $\Delta_1 > \max(\Delta^+, \tilde{\Delta}^+)$ is not true and we can conclude

$$\Delta_1 \leq \max(\Delta^+, \tilde{\Delta}^+). \tag{36}$$

3.1. **Behaviour of Δ^+ in Large Central Charge Limit**

Let us consider Δ^+ as the largest real root of

$$G(\Delta, \mathrm{j}=0) = g^{(N_L)}(\Delta/2+E_0)g^{(N_R)}(\Delta/2+E_0)g^{(N_L')}(E_0)g^{(N_R')}(E_0) - (N_L \leftrightarrow N_L', N_R \leftrightarrow N_R') = 0. \tag{37}$$

In the large central charge limit we can expand it as follows [16]

$$\Delta^+ = \sum_{i=-1} \delta_{-i}\left(\frac{c}{12}\right)^{-i} \tag{38}$$

Let us assume the derivative of order N, is of the order of $\left(\frac{c}{12}\right)^{\alpha_N}$ where $\alpha_N < 1$. From (23) we can show that

$$A^N_{N-n}(i) = N^{2n} + \mathcal{O}(N^{2n-1}). \tag{39}$$

Inserting the expansion of Δ^+ (38), and the definition of E_0 in terms of c, in (22) and using the expansion of $\left(2\pi(\frac{\Delta^+}{2}+\mathrm{E}_0)-\frac{1}{4}-\frac{2\pi\delta_{h,0}}{e^{2\pi}-1}\right)^{N-n}$ in terms of c, we can verify that to leading order in c we have

$$g_N(\frac{\Delta^+}{2}+\mathrm{E}_0) = \sum_{n=0}^{N}(\delta_1-1)^{N-n}\left(\frac{\pi c}{12}\right)^{N-n+2n\alpha_N} + \mathcal{O}\left(\frac{c}{12}\right)^{N-n+2n\alpha_N+\alpha_N-1} \tag{40}$$

For $\alpha_N \geq \frac{1}{2}$, $g_N(\frac{\Delta}{2}+\mathrm{E}_0)$ can be expanded to arbitrary order in c and for $\alpha_N \ll \frac{1}{2}$, $g_N(\frac{\Delta}{2}+\mathrm{E}_0)$ can be expanded to arbitrary order in $\frac{1}{c}$. Therefore, we assume that

$$\alpha_N \ll \frac{1}{2}$$

Plugging (38) to (37), leads to the expansion of $G(\Delta, \mathrm{j}=0)$ to arbitrary order in $\frac{1}{c}$. To leading order in c we have

$$\left(\frac{c}{12}\right)^{N_L+N_R+N_L'+N_R'}\left[-(\delta_1-1)^{N_L+N_R}+(\delta_1-1)^{N_L'+N_R'}\right]=0. \tag{41}$$

The real solutions of this equation are $\delta_1 = 0,1,2$. δ_1 is the largest real root of above equation, so $\delta_1 = 2$. By fixing $\delta_1 = 2$ and keeping the terms in (37) up to order $c^{N_L+N_R+N_L'+N_R'-1}$, δ_0 can be obtained as follows

$$\delta_0=-\frac{(N_L'^2+N_R'^2-N_L{}^2-N_R{}^2)}{2\pi(N_L+N_R-N_L'-N_R')}-\frac{1}{6}-\frac{1}{2\pi}+\frac{2(1+\delta_{\Delta,0})}{e^{2\pi}-1}. \tag{42}$$

Suppose that

$$N_L=\mathcal{O}\left(\frac{c}{12}\right)^{\alpha_N}, \qquad N_R=\mathcal{O}\left(\frac{c}{12}\right)^{\alpha_N},$$
$$N'_L=\mathcal{O}\left(\frac{c}{12}\right)^{\alpha_N}, \qquad N'_R=\mathcal{O}\left(\frac{c}{12}\right)^{\alpha_N}. \tag{43}$$

For obtaining the best upper bound, we must minimize δ_0. The minimum value is obtained in Appendix C. Consequently

$$\Delta^+=\frac{c}{6}-\frac{(N_L'^2+N_R'^2-N_L{}^2-N_R{}^2)}{\pi(N_L+N_R-N_L'-N_R')}-\frac{1}{6}-\frac{1}{2\pi}+\frac{2(1+\delta_{\Delta,0})}{e^{2\pi}-1}. \tag{44}$$

Similarly, in the large central charge limit $\tilde{\Delta}^+$ can also be obtained as follows

$$\tilde{\Delta}^+=\frac{c}{12}+\mathcal{O}(1). \tag{45}$$

Using (45) and (46), an upper bound on Δ_1 can be obtained as follows

$$\Delta_1\le\frac{c}{6}-\frac{(N_L'^2+N_R'^2-N_L{}^2-N_R{}^2)}{\pi(N_L+N_R-N_L'-N_R')}-\frac{1}{6}-\frac{1}{2\pi}+\frac{2(1+\delta_{\Delta,0})}{e^{2\pi}-1}. \tag{46}$$

For example for

$$N_L=20,\quad N_R=21,\quad N_L'=0,\quad N_R'=39 \tag{47}$$

The upper bound in the large central charge limit obtain as follows

$$\Delta_1\le\frac{c}{6}-\frac{340}{\pi}-\frac{1}{6}-\frac{1}{2\pi}+\frac{2(1+\delta_{\Delta,0})}{e^{2\pi}-1}. \tag{48}$$

4. Conclusion

In this paper, we use the medium temperature expansion in order to improve the upper bound on the primary field with lowest dimension. In [23], by using the third and first derivatives in medium temperature expansion an upper bound has been obtained. In order to improve this bound, the linear functional method has been used in two-dimensional CFT with no chiral algebra beyond the Virasoro algebra [19]:

$$\Delta\le\frac{c_{\text{tot}}}{12}-\frac{1}{6}-\frac{1}{2\pi}+\frac{2}{e^{2\pi}-1}.$$

In this paper, we remove the constraint that theory does not have chiral algebra beyond Virasoro algebra and use the medium temperature expansion for an arbitrary order of derivative. Then we obtain the optimal values of the order of derivative which leads to the better upper bound. We obtain an upper bound on Δ_1 as follows

$$\Delta_1\le\frac{c}{6}-\frac{N_L(N_L-3)}{\pi}-\frac{1}{6}-\frac{1}{2\pi}+\frac{2}{e^{2\pi}-1}.$$

where $N_L\approx\left(\frac{c}{12}\right)^{\alpha_N}, \alpha_N\ll\frac{1}{2}$.

As a suggestion for future research, we can also obtain an upper bound on the other primary fields.

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Appendix A

In this Appendix, we obtain (22) and the recursion relation (23).
Using (18), the first derivative of $\chi_h(\tau)$ obtain as follows

$$\tau \frac{\partial}{\partial \tau} \chi_h(\tau) = \tau B_h(\tau) \chi_h(\tau) \tag{49}$$

where

$$B_h(\tau) = 2\pi i(\mathrm{h}+\mathrm{E}_0) - \frac{\eta'(\tau)}{\eta(\tau)} - \frac{2\pi i \delta_{h,0}}{e^{-2\pi i \tau} - 1} \tag{50}$$

It is convenient to write the N th derivative as follows

$$\left(\tau \frac{\partial}{\partial \tau}\right)^N \chi_h(\tau) = \sum_{n=0}^{N} A_n^N(\tau) \left(\tau B_h(\tau)\right)^n \chi_h(\tau) \tag{51}$$

Taking derivative of the (52) gives

$$\left(\tau \frac{\partial}{\partial \tau}\right)^{N+1} \chi_h(\tau) = \sum_{n=0}^{N} \left[\tau \frac{\partial}{\partial \tau} A_n^N(\tau) + n A_n^N(\tau) + (\mathrm{n}+1) A_{n+1}^N(\tau) \tau^2 \frac{\partial}{\partial \tau} B_h(\tau) + A_{n-1}^N(\tau) \right] \left(\tau B_h(\tau)\right)^n \chi_h(\tau) \tag{52}$$

Considering (52) for $N \to N+1$, and comparing it with (53), we can obtain the recursion relation (23). By using[16]

$$\frac{\eta'(\tau)}{\eta(\tau)} = \frac{i}{4} \tag{53}$$

Equation (22) can be obtained.

Appendix B

In this Appendix we obtain the largest real root of $G(\Delta, \mathrm{j})$, for $j \approx \mathcal{O}(\Delta)$. Considering $j = \Delta + p$ where, p is the constant of order one, (31) yields

$$G(\Delta, \Delta + p) = g^{(N_L)}(\Delta + p + E_0) g^{(N_R)}(\mathrm{p} + E_0) g^{(N_L')}(E_0) g^{(N_R')}(E_0) - (N_L \leftrightarrow N_L', N_R \leftrightarrow N_R'). \tag{54}$$

Let us denote the largest real root of (48) by $\bar{\Delta}^+$. In the large central charge limit, we can expand $\bar{\Delta}^+$ as follows

$$\bar{\Delta}^+ = \sum_{a=-1} \bar{\delta}_{-a} \left(\frac{c}{12}\right)^{-a}. \tag{55}$$

Now, we assume $\alpha_N \ll \frac{1}{2}$. Plugging the expansion of $\overline{\Delta}^+$ (56), and the definition of E_0 in terms of c, we can expand $G(\Delta, \Delta+p)$ in terms of $\frac{1}{c}$ and solve it to obtain $\overline{\delta}_{-a}$. Consequently we have the following statement to leading order in c:

$$G(\overline{\Delta}^+, \overline{\Delta}^+ + p) = -\left(\frac{c}{24}\right)^{N_L+N_R+N'_L+N'_R} \left[(2\overline{\delta}_1 - 1)^{N_L} - (2\overline{\delta}_1 - 1)^{N'_L}\right] + \mathcal{O}\left(\frac{c}{24}\right)^{N_L+N_R+N'_L+N'_R-1}. \tag{56}$$

The solutions of $\left[(2\overline{\delta}_1 - 1)^{N_L} - (2\overline{\delta}_1 - 1)^{N'_L}\right] = 0$ are $\overline{\delta}_1 = 0,1$. The largest root of this equation is $\overline{\delta}_1 = 1$. Consequently,

$$\overline{\Delta}^+ = \frac{c}{12} + \mathcal{O}(1). \tag{57}$$

Comparing (58) and (45) one can show that $\overline{\Delta}^+ > \Delta^+$.

Appendix C

In this Appendix we calculate the maximum of $\mathfrak{b}_0$:

$$\mathfrak{b}_0 = \frac{(N'^2_L + N'^2_R - N_L{}^2 - N_R{}^2)}{(N_L + N_R - N'_L - N'_R)}. \tag{58}$$

For this purpose, we assume that N_L, N_R, N'_L, N'_R are continuous real variables. In order to obtain maximum value of $\mathfrak{b}_0$, first we evaluate the gradient of it and set it equal to zero. It is easy to show that it has no local maximum. So, for the nonnegative integer N_L, N_R, N'_L, N'_R, there is no local maximum . For obtaining the maximum, we use the positivity and constraint on N_L, N_R, N'_L, N'_R .

For simplicity we introduce the variable as follow

$$N_L{}^2 + N_R{}^2 = 2a^2 + 1, \qquad N_L + N_R = 2b + 1, \tag{59}$$

$$N'^2_L + N'^2_R = 2e^2 + 1, \qquad N'_L + N'_R = 2d + 1. \tag{60}$$

In order to obtain upper bound on N_L and N_R, we consider the $N_L - N_R$ plane and consider the circle $N_L{}^2 + N_R{}^2 = 2a^2 + 1$ and the line $N_L + N_R = 2b + 1$ in this plane. Since N_L, N_R are positive, from intersection of the line and the circle, we can obtain the bound on a as

$$2b^2 + 2b + 1 \le 2a^2 + 1 \le (2b+1)^2. \tag{61}$$

Similarly, we can obtain the bound on e as follows

$$2d^2 + 2d + 1 \le 2e^2 + 1 \le (2d+1)^2. \tag{62}$$

The maximum of $\mathfrak{b}_0$ occurs at minimum of $|\mathrm{b} - d|$ and maximum of $|a^2 - e^2|$ The minimun of .
$|\mathrm{b} - d|$ is equals to 1 and from (62) and (63) the maximum of the numerator is obtained.
Consequently,

$$\max \mathfrak{b}_0 = b(\mathrm{b} - 3) = N_L(N_L - 3). \tag{63}$$

Which corresponds to

$$N_R = N_L + 1, \qquad N'_R = 2N_L - 1, \qquad N'_L = 0. \tag{64}$$

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بهبود کران بالا روی بعد همدیس میدان‌های اولیه

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چکیده: یکی از ویژگی‌های نظریه میدان همدیس در دو بعد ناوردایی تابع پارش تحت تبدیل‌های آجری است. با به‌کارگیری ناوردایی آجریِ تابع پارش، مجموعه‌ای از قیود روی مشتقات تابع پارش (دریک نقطه‌ی خاص) به‌دست آمده است. این مجموعه از قیود بسط دمای متوسط نامیده می‌شود.

اخیراً، برای نظریه‌هایی که بارمرکزی چپ و بارمرکزی راست در آن‌ها بزرگتر از یک است، با استفاده از بسط دمای متوسط برای مشتق‌های مرتبه اول و مرتبه سوم، کران بالا روی بعد همدیس میدان‌های اولیه ، به‌دست آمده است. به منظور بهبود این کران بالا، روش تابعک خطی در نظریه‌هایی که تنها تقارن دستیده در آن‌ها، تقارن ویراسورو است، به کار گرفته شده است.

در این مقاله با برداشتن این قید که تقارن ویراسورو تنها تقارن دستیده در نظریه باشد، و به‌کارگیری بسط دمای متوسط با درنظر گرفتن مرتبه‌ی دلخواه مشتق‌گیری، کران بهتری روی بُعد همدیس اولین میدان اولیه به‌دست می‌آوریم. نشان خواهیم داد که کران بالا به مرتبه‌ی مشتق‌گیری بستگی دارد. در این مقاله به مطالعه‌ی مقدار بهینه‌ی مرتبه‌ی مشتق‌گیری که توسط آن کران بالای بهتری به‌دست می‌آید، می‌پردازیم.

کلمات کلیدی: نظریه میدان همدیس، ناوردایی آجری، بسط دمای متوسط، میدان اولیه