IRANIAN JOURNAL OF PHYSICS RESEARCH

Iranian Journal of Physics Research, Vol. 17, No. 4, 2017

Improving the upper bound on the conformal dimension in primary fields

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(Received 27 August 2016 ; in final form 6 November 2016)

Abstract
Modular invarinat, constraints the spectrum of the theory. Using the medum temprature expansion, for first and third order of derivative, a universal upper bound on the lowest primary field has been obtained in recent researches. In this paper, we will improve the upper bound, on the scaling dimension of the lowest primary field. We use by the medium temprature expansion for an arbitrary order of derivatives. We show that the upper bound depends on the order of derivative. In this research, we obtain the optimal values of the order of derivatives which leads to the best upper bound.

Keywords: Conformal Field Theory (CFT), modular invariant, medium temprature expansion

1. Introduction

One of the important issues in Conformal Field Theory (CFT) is fixing the theory without relying on the Lagrangian. This is the subject of Bootstrap project [1, 2, 3]. Using the constraints and symmetries which are imposed on the theory, the universal feature will be revealed.

Crossing symmetry is one of the constraints imposed on CFT. By decomposition of the four-point function into the conformal block [4, 5], and using the crossing symmetry in four dimensional CFT, an upper bound on the weights of the fields has been obtained [6-10], that appears in the operator product expansion of two scalar operators. Similarly, a lower bound on the stress tensor central charge has been obtained [11, 12].

In two-dimension, beside the crossing symmetry, the modular invariance is also a powerful constraint that helps us to know much more about the density of states and the spectrum of the theory. The disconnected diffeomorphism group of the torus is a modular group $\mathbf{PSL}(2,\mathbb{Z})$:

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{PSL}(2,\mathbb{Z}), \tag{1}$$

where $\tau = \tau_1 + i\tau_2$ is the complex structure which lies in the upper half plane $(\tau_2 > 0)$, and $\bar{\tau} = \tau_1 - i\tau_2$. The generators of the modular group are $T := (\tau, \bar{\tau}) \to (\tau + 1, \bar{\tau} + 1)$, and $S := (\tau, \bar{\tau}) \to \left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right)$:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

Since the partition function depends only on the conformal structure τ and $\bar{\tau}$; therefore, it must be invariant under modular transformations. From invariance of partition function under T transformation, we can see that the difference between left and right conformal dimension is integer and the difference between left and right central charges is multiple of 24 . The invariance under S transformation is a more powerful constraint and leads to the set of constraints on the density of states [13-15] and the spectrum of the theory [16-25].

Recently, using the modular invariance of partition function, an upper bound on the scaling dimension of primary fields has been obtained. For holomorphically factorizable partition function with $c_L, c_R \in 24\mathbb{Z}$, the holomorphic and antiholomorphic partition functions are modular invariant. In this class of CFT, the lowest

primary field is left moving or right moving, bounded from above as follows:

$$\Delta \le \min\left(\frac{c_L}{24}+1, \frac{c_R}{24}+1\right). \quad (3)$$

In Hellerman [16] study, considering S invariance of partition function at the self-dual point $\tau = -\frac{1}{\tau} = i$, an interesting set of constraints on the partition function has been obtained. In Hellerman [16], by considering the neighborhood of $\tau = -\overline{\tau} = i$

$$\tau \equiv ie^{s}, \quad \overline{\tau} \equiv -ie^{-s} \quad , \quad (4)$$

and taking the derivatives of the S invariance constraint of partition function at $s = 0$:

$$\left(\frac{\partial}{\partial s}\right)^{N_R}\left(\frac{\partial}{\partial \overline{s}}\right)^{N_L} Z(\mathrm{ie}^{s}, -\mathrm{ie}^{\overline{s}})\Big|_{s=0} = \left(\frac{\partial}{\partial s}\right)^{N_R}\left(\frac{\partial}{\partial \overline{s}}\right)^{N_L} Z(\mathrm{ie}^{-s}, -\mathrm{ie}^{-\overline{s}})\Big|_{s=0} \quad , \quad (5)$$

a set of constraints on the partition function is obtained as follows

$$\left(\tau\frac{\partial}{\partial \tau}\right)^{N_R}\left(\overline{\tau}\frac{\partial}{\partial \overline{\tau}}\right)^{N_L} Z(\tau,\overline{\tau})\Big|_{\tau=-\overline{\tau}=i} = 0 \quad , \quad (6)$$

for

$$N_L + N_R = \text{odd} \quad ,$$

which is called medium temperature expansion.

Using the medium temperature expansion in CFT with $c_L, c_R > 1$, an upper bound on the lowest primary field is obtained as follows [24]

$$\Delta \le \frac{c_{\text{tot}}}{12} + 0.4755, \quad c_{\text{tot}} := c_L + c_R. \quad (7)$$

It was shown that for any primary fields with conformal dimension Δ_n with $n \le e^{\frac{\pi c_{tot}}{12}}$, the similar bound has been obtained in the large central charge limit [22]. The bound in the holomorphically factorizable case (5), is a factor of two lower bounds than the bound in the general case (6). Similar to the holomorphically factrorizable case, in special class of (2, 2) supersymmetric theories in the large central charge limit, the similar bound has been obtained [26]. Therefore, one suspects that the bound (6) can be improved. Using the medium temperature expansion method and ST invariance of partition function, it was shown that the upper bound on the primary fields with even spins has been improved by a factor of 2 [23].

The linear functional method in the large central charge limit is used in order to improve the bound (6) as follows

$$\Delta \le \frac{c_{\text{tot}}}{12} - \frac{1}{6} - \frac{1}{2\pi} + \frac{2}{e^{2\pi}-1}. \quad (8)$$

In obtaining the bounds (6) and (7), the first and third order derivatives of partition function have been applied in the canonical ensemble.

In this work, in order to improve the upper bound, we used the medium temperature expansion in the grand canonical ensemble for an arbitrary value of N_L, N_R. Using this constraint, we showed that by increasing the order of derivative, a better upper bound is obtained. However, the order of derivative cannot be increased arbitrarily. There are some constraints about derivatives. We obtained the optimal values of the derivatives, which led to the better upper bound:

$$\Delta_1 \le \frac{c_{tot}}{12} - \frac{(N_L'^2 + N_R'^2 - N_L^2 - N_R^2)}{\pi(N_L + N_R - N_L' - N_R')} - \frac{1}{6} - \frac{1}{2\pi} + \frac{2}{e^{2\pi}-1}. \quad (9)$$

This paper is organized as follows. In section 2, we investigate the partition function of the CFTs with $c_L, c_R > 1$. In section 3, we use the medium temperature expansion for an arbitrary value of N_L, N_R in the grand canonical ensemble. We obtain the optimal values of the derivatives, which leads to a better upper bound. In section 4, we review our results and draw our conclusions. Finally, in appendices, we presented the details of some computations.

2. Decomposition of partition function

The partition function of a general unitary two-dimensional CFT on a torus with complex structure $\tau = \tau_1 + i\tau_2$ can be written as follows

$$Z(\tau,\overline{\tau}) = Tr(e^{2i\pi\tau(L_0 - \frac{c_L}{24})} e^{-2i\pi\overline{\tau}(\overline{L_0} - \frac{c_R}{24})}) = \sum_{h,\overline{h}=0} \rho(h,\overline{h}) e^{2i\pi\tau(h - \frac{c_L}{24})} e^{-2i\pi\overline{\tau}(\overline{h} - \frac{c_R}{24})}, \quad (10)$$

where c_L and c_R are left and right central charges respectively. L_0 and $\overline{L}_0$ are zeroth left-moving and right-moving operators with eigenvalues h and $\overline{h}$, which satisfy Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c_L}{12}(m^3 - m)\delta_{m,-n}, \quad (11)$$

$$[\overline{L}_m, \overline{L}_n] = (m-n)\overline{L}_{m+n} + \frac{c_R}{12}(m^3 - m)\delta_{m,-n}. \quad (12)$$

The Hilbert space of CFT_2 state is characterized by the weight of the primary fields of the theory. The effect of L_n with $n < 0$ on the highest weight state (which corresponds to primary operator) creates descendants of a primary field. For $c_L > 1$ and $h \ne 0$, and for any set of n_i with $n_1 \ge n_2 \ge \cdots \ge n_k \ge 1$, these states are linearly independent:

$$L_{-n_1}L_{-n_2}\cdots L_{-n_k}|0\rangle. \tag{13}$$

For $c_{_L}>1$, and $h=0$, the linearly independent states are given by

$$L_{-n_1}L_{-n_2}\cdots L_{-n_k}|0\rangle \quad , \tag{14}$$

where $n_{_1}\geq n_{_2}\geq\cdots\geq n_{_k}\geq 2$. Therefore, in the partition function, we separate the contribution of identity from other primary operators

$$Z(\tau,\bar{\tau})=Z^{id}(\tau,\bar{\tau})+\sum_A Z^A(\tau,\bar{\tau}), \tag{15}$$

where the sum is over all the primary operators except the identity operator. For CFT$_2$ with $c_L,c_R>1$, the partition function can be written in terms of Virasoro character

$$Z^{(id)}(\tau,\bar{\tau})=\chi_0(\tau)\bar{\chi}_0(\bar{\tau}) \quad , \tag{16}$$

$$Z^{A}(\tau,\bar{\tau})=\chi_{h_A}(\tau)\bar{\chi}_{\bar{h}_A}(\bar{\tau}) \quad , \tag{17}$$

with

$$\chi_h(\tau)\bar{\chi}_{\bar{h}}(\bar{\tau})=\begin{cases}\dfrac{(1-\bar{q})(1-q)}{|\eta(\tau)|^2}\bar{q}^{-\bar{E}_0}q^{-E_0} & \bar{h}=0,h=0,\\ \dfrac{(1-q)}{|\eta(\tau)|^2}\bar{q}^{\bar{h}-\bar{E}_0}q^{h-E_0} & \bar{h}>0,h=0,\\ \dfrac{(1-\bar{q})}{|\eta(\tau)|^2}\bar{q}^{\bar{h}-\bar{E}_0}q^{h-E_0} & \bar{h}=0,h>0,\\ \dfrac{1}{|\eta(\tau)|^2}\bar{q}^{\bar{h}-\bar{E}_0}q^{h-E_0} & \bar{h}>0,h>0.\end{cases} \tag{18}$$

Therefore, we can write the characters as follows

$$\begin{aligned}\chi_h(\tau)&=\frac{q^{h+E_0}}{\eta(\tau)}(1-\mathrm{q})^{\delta_{h,0}},\\ \chi_0(\tau)&=(1-q)\frac{q^{E_0}}{\eta(\tau)},\end{aligned} \tag{19}$$

where $q=e^{2\pi i\tau}$, $E_0=\dfrac{1-c_L}{24}$, $\bar{E}_0=\dfrac{1-c_R}{24}$ and $\eta(\tau)$ is Dedekind eta function.

3. Universal upper bound on the lowest primary field

Using the decomposition of partition function (eq. 14), we rewrite the medium temperature expansion (eq. 5), in the useful configuration as follows

$$\left.\frac{\left(\tau\frac{\partial}{\partial\tau}\right)^{N_L}\left(\bar{\tau}\frac{\partial}{\partial\bar{\tau}}\right)^{N_R}\sum_A Z^A(\tau,\bar{\tau})}{\left(\tau\frac{\partial}{\partial\tau}\right)^{N'_L}\left(\bar{\tau}\frac{\partial}{\partial\bar{\tau}}\right)^{N'_R}\sum_A Z^A(\tau,\bar{\tau})}\right|_{\tau=+i}= \left.\frac{\left(\tau\frac{\partial}{\partial\tau}\right)^{N_L}\left(\bar{\tau}\frac{\partial}{\partial\bar{\tau}}\right)^{N_R}Z^{(id)}(\tau,\bar{\tau})}{\left(\tau\frac{\partial}{\partial\tau}\right)^{N'_L}\left(\bar{\tau}\frac{\partial}{\partial\bar{\tau}}\right)^{N'_R}Z^{(id)}(\tau,\bar{\tau})}\right|_{\tau=+i}. \tag{20}$$

In order to use this constraint, first the derivatives of $Z^{\mathrm{A}}(\tau,\bar{\tau})$ are taken as:

$$\left(\tau\frac{\partial}{\partial\tau}\right)^N Z^A(\tau,\bar{\tau})\Big|_{\tau=i}= (-1)^N g^{(N)}(h_A+\mathrm{E}_0)\chi_{h_A}(i)\bar{\chi}_{\bar{h}_A}(-i), \tag{21}$$

where $g^{(N)}(h)$ is a polynomial, which is defined as follows

$$\left(\tau\frac{\partial}{\partial\tau}\right)^N \chi_h(\tau)\Big|_{\tau=i}=(-1)^N g^{(N)}(h+E_0)\chi_h(i). \tag{22}$$

In what follows, it will be convenient to write the polynomial $g^{(N)}(h)$ as follows

$$g^{(N)}(h)=\sum_{n=0}^{N}(-1)^n A^{(N)}_{N-n}(i)\left(2\pi h-\frac{1}{4}-\frac{2\pi\delta_{h,0}}{e^{2\pi}-1}\right)^{N-n}, \tag{23}$$

where $A^{(N)}_N(i)=1$, and the other $A^{(N)}_N(i)$s can be obtained from the recursion formula as follows

$$\begin{aligned}A^{(N)}_n(\tau)=&\tau\frac{\partial A^{(N-1)}_n}{\partial\tau}-(n+1)\\ &\left(\tau^2\frac{\partial}{\partial\tau}\frac{\eta'(\tau)}{\eta(\tau)}+\frac{(2\pi i\tau)^2e^{-2\pi i\tau}}{e^{-2\pi i\tau}-1}\right)\\ &A^{(N-1)}_{n+1}+nA^{(N-1)}_n+A^{(N-1)}_{n-1}.\end{aligned} \tag{24}$$

The details of the calculation are mentioned in Appendix A. It is easy to solve these recursion formula. $A^{(N)}_{N-1}(\tau)$ and $A^{(N)}_{N-2}(\tau)$ are given by

$$\begin{aligned}A^{(N)}_{N-1}(\tau)&=\frac{N(N-1)}{2},\\ A^{(N)}_{N-2}(\tau)&=\frac{(N-1)^2(N-2)^2}{2\times4}+\frac{(N-1)(N-2)(2N-3)}{2\times6}\\ &\quad-\frac{N(N-1)}{2}\left(\tau^2\frac{\partial}{\partial\tau}\frac{\eta'(\tau)}{\eta(\tau)}+\frac{(2\pi i\tau)^2e^{-2\pi i\tau}}{e^{-2\pi i\tau}-1}\right).\end{aligned} \tag{25}$$

Using eqs. (15) and (20), one can take the derivative of $Z^{(id)}(\tau,\bar{\tau})$:

$$\left(\tau\frac{\partial}{\partial\tau}\right)^N Z^{id}(\tau,\bar{\tau})\Big|_{\tau=i}= (-1)^N g^{(N)}(E_0)\chi_0(i)\overline{\chi_0(-i)}. \tag{26}$$

Inserting eqs. (20) and (25) into eq. (19) yields:

$$\begin{aligned}&\frac{\sum_{A=1}^{\infty}g^{(N_L)}(h_A+E_0)g^{(N_R)}(\bar{h}_A+\bar{E}_0)\Lambda_A e^{-2\pi\Delta_A}}{\sum_{B=1}^{\infty}g^{(N'_L)}(h_B+E_0)g^{(N'_R)}(\bar{h}_B+\bar{E}_0)\Lambda_B e^{-2\pi\Delta_B}}\\ &=\frac{g^{(N_L)}(E_0)g^{(N_R)}(\bar{E}_0)}{g^{(N'_L)}(E_0)g^{(N'_R)}(\bar{E}_0)}:=G_0(E_0,\bar{E}_0),\end{aligned} \tag{27}$$

where,

$$\Lambda_A = (1-e^{2\pi})^{\delta_{h_A,0}+\delta_{\bar{h}_A,0}}. \quad (28)$$

Subtracting both sides of eq. (26) results to

$$\frac{\Sigma\Big[g^{(N_L)}(h_A+E_0)g^{(N_R)}(\bar{h}_A+\bar{E}_0)-G_0(E_0,\bar{E}_0)}{\sum_{B=1}^{\infty} g^{(N'_L)}(h_B+E_0)g^{(N'_R)}(\bar{h}_B+\bar{E}_0)\Lambda_B e^{-2\pi\Delta_B}}$$

$$\frac{g^{(N'_L)}(h_A+E_0)g^{(N'_R)}(\bar{h}_A+\bar{E}_0)\Big]\Lambda_A e^{-2\pi\Delta_A}}{\sum_{B=1}^{\infty} g^{(N'_L)}(h_B+E_0)g^{(N'_R)}(\bar{h}_B+\bar{E}_0)\Lambda_B e^{-2\pi\Delta_B}}=0. \quad (29)$$

In the following, we assume $c_L = c_R := c$. In order to proceed, we write the conformal weight h and $\bar{h}$ in terms of the scaling dimension Δ, and the spin j:

$$\Delta = h+\bar{h},$$
$$j = h-\bar{h}. \quad (30)$$

Therefore, eq. (28) can be written as follows

$$\frac{\sum_{A=1}^{\infty} G(\Delta_A, j_A)\Lambda_A e^{-2\pi\Delta_A}}{g^{N_L}(E_0)g^{N'_R}(E_0)\sum_B g^{N'_L}} \quad (31)$$

$$\frac{\sum_{A=1}^{\infty} G(\Delta_A, j_A)\Lambda_A e^{-2\pi\Delta_A}}{\left(\frac{\Delta+j}{2}+E_0\right)g^{N'_R}\left(\frac{\Delta-j}{2}+E_0\right)\Lambda_B e^{-2\pi\Delta_B}}=0,$$

where,

$$G(\Delta, \mathrm{j}) = g^{(N_L)}((\Delta+j)/2+E_0)g^{(N_R)} ((\Delta-j)/2+E_0)g^{(N'_L)}(E_0)g^{(N'_R)}(E_0) - (N_L \leftrightarrow N'_L, N_R \leftrightarrow N'_R). \quad (32)$$

Since $G(\Delta, \mathrm{j})$ is an odd function of Δ, it has at least one real root. $G(\Delta, \mathrm{j})$ is a function of Δ and j. In unitary CFT, from $h \geq o$ and $\bar{h} \geq o$, we conclude $-\Delta \leq j \leq \Delta$. Therefore, the roots of $G(\Delta, \mathrm{j})$ depends on the value of spin. Let us denote the largest real root of $G(\Delta, \mathrm{j})$ for $j \approx \mathcal{O}(1)$ by Δ^+ and for $j \approx \mathcal{O}(\Delta)$ by $\bar{\Delta}^+$. In the limit of $\Delta \to \infty$, $G(\Delta, \mathrm{j})$ goes to infinity, therefore, for $\Delta > \max(\Delta^+, \bar{\Delta}^+)$, the function $G(\Delta, \mathrm{j})$ is positive. We show in Appendix B that $\max(\Delta^+, \bar{\Delta}^+) = \Delta^+$. Therefore, for $\Delta > \Delta^+$, $G(\Delta, \mathrm{j})$ is positive.

Now, suppose that in eq. (30):

$$0 = \Delta_0 < \Delta_1 \leq \Delta_2 \leq \Delta_3 \leq \cdots. \quad (33)$$

Hence, from

$$\Delta_n \geq \Delta_1 > \Delta^+, \quad \text{all n} > 0, \quad (34)$$

we verify that

$$G(\Delta_n, \mathrm{j}_n) > 0, \quad \text{all } n > 0. \quad (35)$$

Now, suppose that $\tilde{\Delta}^+$ is the largest real root of $g^{N'_L}\left(\frac{\Delta+j}{2}+E_0\right)g^{N'_R}\left(\frac{\Delta-j}{2}+E_0\right)=0$. Similarly, for

$$\Delta_n \geq \Delta_1 > \tilde{\Delta}^+, \quad \text{all n} > 0, \quad (36)$$

we have

$$g^{N'_L}\left(\frac{\Delta_n+j_n}{2}+E_0\right)g^{N'_R}\left(\frac{\Delta_n-j_n}{2}+E_0\right)>0 \quad \text{all } n>0. \quad (37)$$

Consequently, for $\Delta_1 > \max(\Delta^+, \tilde{\Delta}^+)$, every terms in the numerator and denominator of the right hand side of eq. (30) are positive. It is in contrast with the left hand side of eq. (30). Thus, the hypothesis $\Delta_1 > \max(\Delta^+, \tilde{\Delta}^+)$ is not true and we can conclude that

$$\Delta_1 \leq \max(\Delta^+, \tilde{\Delta}^+). \quad (38)$$

3.1. Behaviour of Δ^+ in large central charge limit

Let us consider Δ^+ as the largest real root of

$$G(\Delta, \mathrm{j}=0) = g^{(N_L)}(\Delta/2+E_0)g^{(N_R)} (\Delta/2+E_0)g^{(N'_L)}(E_0)g^{(N'_R)}(E_0) - (N_L \leftrightarrow N'_L, N_R \leftrightarrow N'_R) = 0. \quad (39)$$

In the large central charge limit, we can expand it as follows [16]

$$\Delta^+ = \sum_{i=-1} \delta_{-i}\left(\frac{c}{12}\right)^{-i}. \quad (40)$$

Let us assume that the derivative of order N, is of the order of $\left(\frac{c}{12}\right)^{\alpha_N}$, where $\alpha_N < 1$. From eq. (23) we can show that

$$A^N_{N-n}(i) = N^{2n} + \mathcal{O}(N^{2n-1}). \quad (41)$$

Inserting the expansion of Δ^+ (eq. 38), and based on the definition of E_0 in terms of c in eq. (22) as well as using the expansion of $\left(2\pi(\frac{\Delta^+}{2}+\mathrm{E}_0)-\frac{1}{4}-\frac{2\pi\delta_{h,0}}{e^{2\pi}-1}\right)^{N-n}$ in terms of c, we can verify that to leading order in c we have

$$g_N(\frac{\Delta^+}{2}+\mathrm{E}_0) = \sum_{n=0}^{N}(\delta_1-1)^{N-n}\left(\frac{\pi c}{12}\right)^{N-n+2n\alpha_N} + \mathcal{O}\left(\frac{c}{12}\right)^{N-n+2n\alpha_N+\alpha_N-1}. \quad (42)$$

For $\alpha_N \geq \frac{1}{2}$, $g_N(\frac{\Delta}{2}+\mathrm{E}_0)$ can be expanded to arbitrary order in c and for $\alpha_N \ll \frac{1}{2}$, $g_N(\frac{\Delta}{2}+\mathrm{E}_0)$ can be expanded to arbitrary order in $\frac{1}{c}$. Therefore, we assume that

$$\alpha_N \ll \frac{1}{2} . \tag{43}$$

Plugging eq. (38) to eq. (37) leads to the expansion of $G(\Delta, \mathrm{j}=0)$ to arbitrary order in $\frac{1}{c}$. To leading order in c we have

$$\left(\frac{c}{12}\right)^{N_L+N_R+N'_L+N'_R} \left[-(\delta_1-1)^{N_L+N_R}+(\delta_1-1)^{N'_L+N'_R}\right]=0. \tag{44}$$

The real solutions of this equation are $\delta_1 = 0,1,2$. δ_1 is the largest real root of the above equation, so $\delta_1 = 2$. By fixing $\delta_1 = 2$ and keeping the terms in eq. (37) up to order $c^{N_L+N_R+N'_L+N'_R-1}$, δ_0 can be obtained as follows

$$\delta_0 = -\frac{({N'_L}^2+{N'_R}^2-{N_L}^2-{N_R}^2)}{2\pi(N_L+N_R-N'_L-N'_R)}-\frac{1}{6} -\frac{1}{2\pi}+\frac{2(1+\delta_{\Delta,0})}{e^{2\pi}-1}. \tag{45}$$

Suppose that

$$N_L = \mathcal{O}\left(\frac{c}{12}\right)^{\alpha_N}, \quad N_R = \mathcal{O}\left(\frac{c}{12}\right)^{\alpha_N}, \quad N'_L = \mathcal{O}\left(\frac{c}{12}\right)^{\alpha_N}, \quad N'_R = \mathcal{O}\left(\frac{c}{12}\right)^{\alpha_N}. \tag{46}$$

For obtaining the best upper bound, we must minimize δ_0. The minimum value is obtained in Appendix C. Consequently

$$\Delta^+ = \frac{c}{6}-\frac{({N'_L}^2+{N'_R}^2-{N_L}^2-{N_R}^2)}{\pi(N_L+N_R-N'_L-N'_R)}-\frac{1}{6} -\frac{1}{2\pi}+\frac{2(1+\delta_{\Delta,0})}{e^{2\pi}-1}. \tag{47}$$

Similarly, in the large central charge limit, $\tilde{\Delta}^+$ can also be obtained as follows

$$\tilde{\Delta}^+ = \frac{c}{12}+\mathcal{O}(1). \tag{48}$$

Using eqs. (45) and (46), an upper bound on Δ_1 can be obtained as follows

$$\Delta_1 \leq \frac{c}{6}-\frac{({N'_L}^2+{N'_R}^2-{N_L}^2-{N_R}^2)}{\pi(N_L+N_R-N'_L-N'_R)}-\frac{1}{6} -\frac{1}{2\pi}+\frac{2(1+\delta_{\Delta,0})}{e^{2\pi}-1}. \tag{49}$$

For example for

$$N_L = 20, \quad N_R = 21, \quad N'_L = 0, \quad N'_R = 39 , \tag{50}$$

the upper bound in the large central charge limit is obtained as follows

$$\Delta_1 \leq \frac{c}{6}-\frac{340}{\pi}-\frac{1}{6}-\frac{1}{2\pi}+\frac{2(1+\delta_{\Delta,0})}{e^{2\pi}-1}. \tag{51}$$

4. Conclusion

In this paper, we used the medium temperature expansion in order to improve the upper bound on the primary field with lowest dimension. In Qualls' [23] study, by using the third and the first derivatives in medium temperature expansion, an upper bound was obtained. In order to improve this bound, the linear functional method was used in two-dimensional CFT with no chiral algebra beyond the Virasoro algebra [19]:

$$\Delta \leq \frac{c_{\mathrm{tot}}}{12}-\frac{1}{6}-\frac{1}{2\pi}+\frac{2}{e^{2\pi}-1}.$$

In this paper, we removed the constraint that the theory does not have chiral algebra beyond Virasoro algebra and used the medium temperature expansion for an arbitrary order of derivative. Then, we obtained the optimal values of the order of derivative which led to a better upper bound. We obtained an upper bound on Δ_1 as follows

$$\Delta_1 \leq \frac{c}{6}-\frac{N_L(N_L-3)}{\pi}-\frac{1}{6}-\frac{1}{2\pi}+\frac{2}{e^{2\pi}-1} \quad ,$$

where

$$N_L \approx \left(\frac{c}{12}\right)^{\alpha_N} , \quad \alpha_N \ll \frac{1}{2} .$$

As a suggestion for future research, one can also obtain an upper bound on the other primary fields.

5. Acknowledgments

This paper is part of my PhD research at Isfahan University of Technology, Iran. I would like to thank my thesis advisor Dr. Farhan Loran for all his help, support and guidance during my PhD research.

5. 1. Appendix A

In this Appendix, we obtain eq. (22) and the recursion relation (eq. 23). Using eq. (18), the first derivative of $\chi_h(\tau)$ was obtained as follows

$$\tau\frac{\partial}{\partial\tau}\chi_h(\tau) = \tau B_h(\tau)\chi_h(\tau) \quad , \tag{52}$$

where

$$B_h(\tau) = 2\pi i(\mathrm{h}+\mathrm{E}_0) - \frac{\eta'(\tau)}{\eta(\tau)} - \frac{2\pi i \delta_{h,0}}{e^{-2\pi i\tau}-1} \quad . \tag{53}$$

It is convenient to write the N^{th} derivative as follows

$$\left(\tau\frac{\partial}{\partial\tau}\right)^N \chi_h(\tau) = \sum_{n=0}^{N} A_n^N(\tau)\left(\tau B_h(\tau)\right)^n \chi_h(\tau) \quad . \tag{54}$$

Taking derivative of eq. (52) gives

$$\left(\tau\frac{\partial}{\partial\tau}\right)^{N+1} \chi_h(\tau) = \sum_{n=0}^{N}\Bigg[\tau\frac{\partial}{\partial\tau}A_n^N(\tau) + nA_n^N(\tau) + (\mathrm{n}+1)A_{n+1}^N \tag{55}$$
$$(\tau)\tau^2\frac{\partial}{\partial\tau}B_h(\tau) + A_{n-1}^N(\tau)\Bigg]\left(\tau B_h(\tau)\right)^n \chi_h(\tau) \quad .$$

Considering eq. (52) for $N \to N+1$, and comparing it with eq. (53), we obtained the recursion relation of eq.(23). By using Hellerman's [16] findings

$$\frac{\eta'(\tau)}{\eta(\tau)} = \frac{i}{4} \quad . \tag{56}$$

Equation (22) was obtained.

5. 2. Appendix B

In this Appendix, we obtain the largest real root of $G(\Delta, \mathrm{j})$, for $j \approx \mathcal{O}(\Delta)$. Considering $j = \Delta + p$ where p is the constant of order one, eq. (31) yields

$$G(\Delta, \Delta+p) = g^{(N_L)}(\Delta+p+E_0)g^{(N_R)} \tag{57}$$
$$(\mathrm{p}+E_0)g^{(N'_L)}(E_0)g^{(N'_R)}(E_0)$$
$$-(N_L \leftrightarrow N'_L, N_R \leftrightarrow N'_R).$$

Let us denote the largest real root of eq. (48) by $\bar{\Delta}^+$. In the large central charge limit, we expanded $\bar{\Delta}^+$ as follows

$$\bar{\Delta}^+ = \sum_{a=-1} \bar{\delta}_{-a}\left(\frac{c}{12}\right)^{-a} . \tag{58}$$

Then, we assumed $\alpha_N \ll \frac{1}{2}$. Plugging the expansion of $\bar{\Delta}^+$ (56), and based on the definition of E_0 in terms of c , we expanded $G(\Delta, \Delta+p)$ in terms of $\frac{1}{c}$ and solved it to obtain $\bar{\delta}_{-a}$. Consequently, we found the following statement to leading order in c :

$$G(\bar{\Delta}^+, \bar{\Delta}^+ + p) = -\left(\frac{c}{24}\right)^{N_L+N_{R+}N'_L+N'_R} \tag{59}$$
$$\left[(2\bar{\delta}_1 - 1)^{N_L} - (2\bar{\delta}_1 - 1)^{N'_L}\right]$$
$$+\mathcal{O}\left(\frac{c}{24}\right)^{N_L+N_{R+}N'_L+N'_{R-1}} .$$

The solutions of $\left[(2\bar{\delta}_1 - 1)^{N_L} - (2\bar{\delta}_1 - 1)^{N'_L}\right] = 0$ are $\bar{\delta}_1 = 0,1$. The largest root of this equation is $\bar{\delta}_1 = 1$. Consequently,

$$\bar{\Delta}^+ = \frac{c}{12} + \mathcal{O}(1). \tag{60}$$

Comparing eq. (58) and eq. (45) one showed that $\bar{\Delta}^+ > \Delta^+$.

5. 3. Appendix C

In this Appendix we calculate the maximum of $\mathfrak{b}_0$:

$$\mathfrak{b}_0 = \frac{(N'_L{}^2 + N'_R{}^2 - N_L{}^2 - N_R{}^2)}{(N_L + N_R - N'_L - N'_R)}. \tag{61}$$

For this purpose, we assumed that N_L, N_R, N'_L, N'_R were continuous real variables. In order to obtain maximum value of $\mathfrak{b}_0$, first, we evaluated the gradient of it and set it equal to zero. It was easy to show that it had no local maximum. So, for the nonnegative integer N_L, N_R, N'_L, N'_R, there was no local maximum. For obtaining the maximum, we used the positivity and constraint on N_L, N_R, N'_L, N'_R .

For simplicity we introduced the variable as follow

$$\begin{aligned} N_L{}^2 + N_R{}^2 &= 2a^2 + 1, \\ N_L + N_R &= 2b + 1, \end{aligned} \tag{62}$$

$$\begin{aligned} N'_L{}^2 + N'_R{}^2 &= 2e^2 + 1, \\ N'_L + N'_R &= 2d + 1. \end{aligned} \tag{63}$$

In order to obtain upper bound on N_L and N_R, we considered the $N_L - N_R$ plane and considered the circle $N_L{}^2 + N_R{}^2 = 2a^2 + 1$ and the line $N_L + N_R = 2b + 1$ in this plane. Since N_L, N_R are positive, from intersection of the line and the circle, we obtained the bound on a as

$$2b^2 + 2b + 1 \le 2a^2 + 1 \le (2b+1)^2. \tag{64}$$

Similarly, we obtained the bound on e as follows

$$2d^2 + 2d + 1 \le 2e^2 + 1 \le (2d+1)^2. \tag{65}$$

The maximum of $\mathfrak{b}_0$ occurred at minimum of $|\mathrm{b}-d|$ and maximum of $|a^2 - e^2|$ .The minimun of $|\mathrm{b}-d|$ was equal to 1 and from eqs. (62) and (63), the maximum of the numerator was obtained. Consequently,

$$\max \mathfrak{b}_0 = b(\mathrm{b}-3) = N_L(N_L - 3). \tag{66}$$

which corresponds to

$$\begin{aligned} N_R &= N_L + 1, \\ N'_R &= 2N_L - 1, \\ N'_L &= 0. \end{aligned} \tag{67}$$

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