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A new symmetry for large deviation functions of time-integrated dynamical variables

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Abstract
A new type of symmetry in the large deviation function of a time-integrated current is introduced. This current is different from the fluctuating entropy production for which the large deviation function is symmetric in the content of the fluctuation theorem. The origin of this symmetry, similar to that of the Gallavotti-Cohen-Evans-Morriss symmetry, is related to time-reversal. The symmetry is more unveiled when one performs an appropriate grouping of stochastic trajectories in the space of microscopic configurations. It turns out that the characteristic polynomial of the modified generator of this current is not symmetric; however, its minimum eigenvalue is symmetric.

Keywords: out of equilibrium systems, fluctuation theorem, large deviation in out of equilibrium systems, symmetries of large deviation function

1. Introduction

Most of the systems in nature are exposed to a flux of matter or energy in the stationary state, and therefore are driven out of equilibrium. These non-equilibrium systems are usually modeled as Markov jump processes. In a Markov jump process the system jumps from one configuration to another configuration in the configuration space with a certain transition rate. These microscopic transition rates do not satisfy the detailed balance; hence the system will relax into a non-equilibrium state where the probability current between configurations is non-zero. In order to preserve this probability current, the system should be driven by an external drive which continuously produces entropy in the environment.

Different fluctuating quantities (or functionals) can be defined which depend on the specific sequence of transitions or a stochastic trajectory in the configuration space. In long-time limit the fluctuation theorems restrict the functional form of the probability distribution of this fluctuating quantity. Application of the large deviation theory reveals that the Gallavotti-Cohen-Evans-Morriss (GCEM) symmetry can be considered as a symmetry of the large deviation function for the probability distribution [1- 4].

It was thought that the entropy produced in the environment was the only time-integrated current (entropic current) for which the fluctuation theorem is valid [5-7]. Recent investigations have shown that a different time-integrated current (non-entropic current) exists which displays a symmetric large deviation function [8-10]. Interestingly this symmetry is slightly different from the GCEM symmetry. The authors in [8] have shown that the height of an interface in a certain growth model is a physically relevant example of a non-entropic time-integrated current with a symmetric large deviation function. Necessary condition in order to have a non-entropic current with the GCEM symmetry in a Markov pure jump process is presented in [10]. In has also been shown that this condition is related to degeneracies in the set of increments associated with fundamental cycles from Schnakenberg network theory [11]. On the other hand, the symmetry is originated in the time-reversal of the appropriately grouped of trajectories [9, 10]. From a mathematical point of view, the Gärtner-Ellis theorem states that the large deviation function of a time-integrated current is given by the Legendre-Fenchel transformation of the minimum eigenvalue of a modified generator as- sociated with the current under consideration [12]. To the best of our knowledge the characteristic polynomials of the modified generators associated with the non-entropic currents which have been studied so far are symmetric. This means that all the eigenvalues of the characteristic polynomials, including the minimum eigenvalue, are symmetric.

A question which has not been answered in the related literature is that whether one can define a non-entropic time-integrated current with a symmetric large deviation function without requiring a symmetric characteristic polynomial of the modified generator. This means that only some of the eigenvalues of the characteristic polynomial, including the minimum eigenvalue, are symmetric.

In this paper we propose a simple network of microscopic transition rates in which a properly defined non-entropic time-integrated current has a symmetric large deviation function; however, the characteristic polynomial of its modified generator is not symmetric. This implies that only the minimum eigenvalue, and not all the eigenvalues, is symmetric. Our exact analytical results besides the numerical investigations show that such a symmetry exists and it is related to the time-reversal of properly grouped stochastic trajectories in the configuration space. Although the example introduced in present paper is very restricted; however, it predicts that this type of symmetry, first mentioned in [10, 9], exists and, at least in our example, it does not belong to the time-reversal of some most probable trajectory in the configuration space. It also turns out that the structure of the configuration space plays an important role.

This paper is organized as follows. The second section is dedicated to a brief review of the basics of the large deviation theory. In the third section we discuss the symmetries of the large deviation functions studied in the literature. We introduce our network and a time-integrated current and study its symmetries in the fourth section. Finally, we explain the origin of the symmetry of this non-entropic time-additive dynamical variable which will be called the current throughout this paper.

2. Basics of large deviation theory

It is known that the theory of large deviations can be applied to study the properties of both equilibrium and non-equilibrium systems [12]. The large deviation theory starts with the observation that the dominant term of the probability distribution of a random variable exponentially decays to zero. In order to briefly review this theory, let us first consider a continuous-time Markov process with a finite configuration space S in which a spontaneous transition from configuration s to configuration s', where $s,s' \in S$, takes place with a transition rate $w_{s \to s'}$. The time evolution of the probability distribution $P(s,t)$, for the system being in s at time t, is given by a master equation

$$\frac{d}{dt}P(s,t) = -\sum_{s'} H_{ss'} P(s',t) \tag{1}$$

where H is the Markov generator with non-diagonal and diagonal elements given by

$$H_{ss'} = -w_{s' \to s}$$

and

$$H_{ss} = \sum_{s' \neq s} w_{s \to s'}$$

where $\lambda_s = H_{ss}$ is usually called the scape rate from the configuration s. Using the quantum Hamiltonian formalism, the master equation (1) can be rewritten as [13]

$$\frac{\mathrm{d}}{\mathrm{dt}}\left|P(s,t)\right\rangle = \mathrm{H}\left|P(s,t)\right\rangle \tag{2}$$

A stochastic trajectory $\vec{S}_{M,t}$ in the configuration space is defined as a sequence of M consecutive jumps $s(\mathrm{t}_0 = 0) \to s(\mathrm{t}_1) \to \cdots \to s(\mathrm{t}_\mathrm{M})$ taking place at chronologically ordered times $\mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_\mathrm{M} \in [0, \mathrm{t}]$ where M is a random variable. On the other hand, the reversed trajectory $\bar{S}_{M,t}$ is defined as a sequence of M consecutive jumps $s(\mathrm{t}_\mathrm{M}) \to s(\mathrm{t}_\mathrm{M} - 1) \to \cdots \to s(\mathrm{t}_0 = 0)$ at times $\mathrm{t} - \mathrm{t}_\mathrm{M}, \mathrm{t} - \mathrm{t}_{\mathrm{M}-1}, \ldots, \mathrm{t} - \mathrm{t}_1 \in [0, \mathrm{t}]$. A time-integrated current $\mathcal{J}$ is a functional of the stochastic trajectory $\vec{S}_{M,t}$ in S during the time t. If the current changes its value by $\theta_{s \to s'}$ whenever a jump from $s \to s'$ occurs, we have

$$\mathcal{J}\left[\vec{S}_{M,t}\right] = \sum_{i=1}^{M} \theta_{s(t_{i-1}) \to s(t_i)}. \tag{3}$$

The increment $\theta_{s \to s'}$ is antisymmetric and in the case $\theta_{s \to s'} = ln\frac{w_{s \to s'}}{w_{s' \to s}}$, $\mathcal{J}$ is just the entropy change. The generating function for $\mathcal{J}$ can be written as [5, 6]

$$\left\langle e^{-\mu \mathcal{J}} \right\rangle = \left\langle \mathrm{I} \left| e^{-\hat{H}t} \right| P_0 \right\rangle \tag{4}$$

in which $\langle I|$ is a summation vector $(1,1,1,...)$ and that $|P_0\rangle$ is the initial probability distribution vector. The non-diagonal and diagonal matrix elements of the modified generator $\hat{H}$ in (4) are also given by [6]

$$\hat{H}_{ss'} = -w_{s' \to s} e^{-\mu\theta_{s' \to s}}$$

and

$$\hat{H}_{ss} = \sum_{s' \neq s} w_{s \to s'}$$

respectively. If S is bounded, in the long-time limit $t \to \infty$ the generating function (4) can be written as [6]

$$\lim_{t \to \infty} \left\langle e^{-\mu \mathcal{J}} \right\rangle = e^{-te(\mu)} \tag{5}$$

In which $e(\mu)$ is the minimum eigenvalue of the modified generator $\hat{H}$.

We expect that in the long-time limit, the quotient $\mathcal{J}/t$ tends to a constant J. If the probability distribution of J satisfies a large deviation principle, then in the long-time limit we can write

$$\lim_{t\to\infty} P(J,t) \sim e^{-t\hat{e}(J)} \quad (6)$$

In which $\hat{e}(J)$ is the large deviation function associated with the time-integrated current $\mathcal{J}$, which is related to deviations of the current from its average value. According to the Gärtner-Ellis theorem, the Legendre transformation of gives the large deviation function of $e(\mu)$ gives the large deviation function [12]

$$\hat{e}(J) = \max_{\mu}\left(e(\mu) - J\mu\right). \quad (7)$$

In the next section we discuss the symmetries of this large deviation function studied in related literature.

3. Symmetries of large deviation function

As we mentioned, the fluctuation relations which are generally classified as finite-time fluctuation relations and infinite-time fluctuation relations, restrict the functional form of the large devotion function. Here we only consider the infinite-time fluctuation relations. Two different types of infinite time fluctuation relations are studied in the context of the Markov jump processes: the GCEM symmetry and what it is called the GCEM-like symmetry. Although these symmetries exert different restrictions on the functional form of the large deviation function, it has been shown that they have identical physical origin which is in fact time-reversal [5- 10].

It is known that the GCEM symmetry can always be considered as the symmetry of the large deviation function for the probability distribution of the entropy and entropic currents [5]; however, it seems that the existence of the GCEM-like symmetry for the probability distribution of the non-entropic currents highly depends on the microscopic transition rates between different configurations in the configuration space [9, 10].

Form a mathematical perspective, both the GCEM symmetry and the GCEM- like symmetry refer to the following relation for the large deviation function of the current J

$$\hat{e}(-J) - \hat{e}(J) = EJ \quad (8)$$

in which E is a field conjugated to the current J. It has been shown that if the fluctuations of the current obey the GCEM symmetry then the conjugate field E can be obtained from [6]

$$\hat{H}^T(\mu) = P_{eq}^{-1}\hat{H}(E-\mu)P_{eq} \quad (9)$$

where T is transpose of a square matrix. P_{eq} is a diagonal matrix with elements which are the equilibrium probabilities of the corresponding undriven system whose transition rates satisfy detailed balance. One should note that (9) also means that all of the eigenvalues of $\hat{H}(\mu)$ and $\hat{H}(E-\mu)$ are identical and symmetric including the minimum eigenvalue which satisfies

$$e(\mu) = e(E-\mu). \quad (10)$$

In other words, the characteristic polynomial of the modified generator $\hat{H}(\mu)$ defined as

$$P(\mu,x) = \det\left(\hat{H}(\mu) - xI\right) \quad (11)$$

in which I is the identity matrix, is symmetric i.e., we have [14, 15]

$$P(\mu,x) = P(E-\mu,x). \quad (12)$$

The entropic currents, for instance the entropy production and those time-integrated currents which are proportional to the entropy production, satisfy the GCEM symmetry and also have a symmetric characteristic polynomial.

The large deviation function of non-entropic currents or those time-integrated currents which are not proportional to the entropy production satisfy (8) without requiring (9) to be satisfied. This type of symmetry is known as the GCEM-like symmetry. A physically relevant example was first introduced in [8] where the height of an interface in a certain growth model was defined as a time-integrated non-entropic current with a symmetric large deviation function. Later this idea was extended to more general Markov jump processes in [9] and [10]. In these papers the authors have obtained the necessary conditions on the microscopic transition rates to have a non-entropic current with a GCEM-like symmetry. It turns out that the characteristic polynomial of the modified generator for these non-entropic currents always satisfy (12). This means that all of the eigenvalues of the modified generator, including the minimum eigenvalue which plays a crucial role in the Gärtner-Ellis theorem, is symmetric.

4. Time reversal as the origin of symmetry

Let us first consider an entropic time-integrated current $\mathcal{J}$ whose large deviation function satisfies the GCEM symmetry. The Schnakenberg relation indicates that if the ratio of the weight of a given cycle $\mathcal{C}$ in the network of states, defined as a product of transition rates of the cycle, to its time-reversal $\bar{\mathcal{C}}$ is given by [11, 14]

$$\frac{W_{\mathcal{C}}}{W_{\bar{\mathcal{C}}}} = e^{EK_{\mathcal{C}}} \quad (13)$$

in which $K_{\mathcal{C}}$ is the increment of the cycle $\mathcal{C}$, then in the long-time limit the ration of the weight of a stochastic trajectory $\vec{S}_{M,t}$ to the weight of its time reversed $\bar{S}_{M,t}$ is given by

$$\frac{W\left[\vec{S}_{M,t}\right]}{W\left[\bar{S}_{M,t}\right]} = e^{E\mathcal{J}\left[\vec{S}_{M,t}\right]}. \quad (14)$$

In the long-time limit this leads us to the fluctuation theorem

$$\frac{P(\mathcal{J})}{P(-\mathcal{J})} = e^{E\mathcal{J}}. \quad (15)$$

For a non-entropic time-integrated current whose large deviation function satisfies a GCEM-like symmetry, one has [9, 10]

$$\frac{W_{\{\mathcal{C}\}}}{W_{\{\bar{\mathcal{C}}\}}} = e^{EK_{\{\mathcal{C}\}}} \quad (16)$$

in which $\{\mathcal{C}\}$ indicates a group of cycles with equal

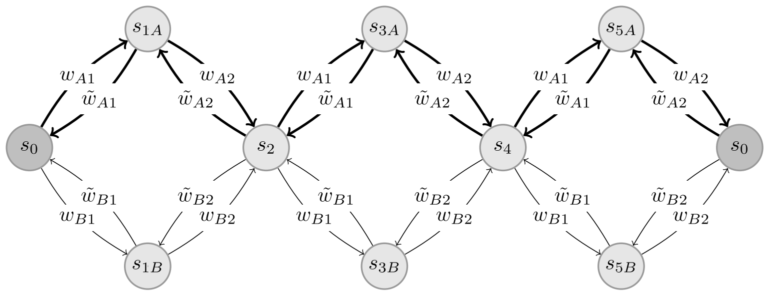

Figure 1. A network of states with periodic boundary conditions. The bold lines show the links where the current $\mathcal{J}_A$ is defined (see inside the text).

increments. A symmetric characteristic polynomial requires that all of the cycles with equal increments satisfy (16) with a given E associated with the current. Considering a group of trajectories $\{\vec{S}_{M,t}\}$ belonging to the same class one finds

$$\frac{W\left[\{\vec{S}_{M,t}\}\right]}{W\left[\{\vec{S}_{M,t}\}\right]} = e^{E\mathcal{J}\left[\{\vec{S}_{M,t}\}\right]} \tag{17}$$

in which $W\left[\{\vec{S}_{M,t}\}\right]$ indicates the sum of the weights of the trajectories of the group. Similarly, this leads us to the fluctuation theorem (15) [8, 9, 10].

We might ask whether we can define a configuration space and a non-entropic current with a large deviation function which satisfies (8) without requiring (9) and (12) to be satisfied. In this case only the minimum eigenvalue of the characteristic polynomial of the modified generator of this current will be symmetric. In the next section we provide a simple finite state space with restricted microscopic transition rates and show that one can actually define a time-integrated current with this type of symmetry.

5. Definition of a new symmetric current

Let us consider a finite configuration space S as a graph as it is shown in Figure 1. The vertices of this graph are the configurations or the states of the system and the edges represent the possible transitions between those states. As can be seen this network of configurations has a periodic structure. The forward (rightward) and backward (leftward) transition rates between different configurations are denoted by w and $\tilde{w}$ respectively.

The steady-state of a system with this configuration space can be easily obtained. Considering the symmetry properties of the network, the steady-state probability for being in different configurations are given by

$$\begin{aligned} P_{s_0,s_2,s_4} &= \frac{1}{Z}(\tilde{w}_{A1}+w_{A2})(\tilde{w}_{B1}+w_{B2}), \\ P_{s_{1A},s_{3A},s_{5A}} &= \frac{1}{Z}(\tilde{w}_{A2}+w_{A1})(\tilde{w}_{B1}+w_{B2}), \\ P_{s_{1B},s_{3B},s_{5B}} &= \frac{1}{Z}(\tilde{w}_{A1}+w_{A2})(\tilde{w}_{B2}+w_{B1}), \end{aligned} \tag{18}$$

where the normalization factor Z is

$$Z = 3\begin{pmatrix} (\tilde{w}_{A1}+w_{A2})(\tilde{w}_{B1}+w_{B2}) \\ +(\tilde{w}_{A2}+w_{A1})(\tilde{w}_{B1}+w_{B2}) \\ +(\tilde{w}_{A1}+w_{A2})(\tilde{w}_{B2}+w_{B1}) \end{pmatrix}$$

Now we define a time-integrated current $\mathcal{J}_A$ of type (3), with $\theta_{s\to s'}=1$ and $\theta_{s\to s'}=-1$ for a forward and a backward transition respectively, which goes through the following links

$$s \to s_{1A} \to s_2 \to s_{3A} \to s_4 \to s_{5A} \to s_0$$

as we have shown in Figure 1 in bold. Using the steady-state probabilities (18) the long-time average of the current $J_A = \mathcal{J}_A / t$ can be calculated

$$J_A = \frac{1}{Z}\left(6(w_{A1}w_{A2}-\tilde{w}_{A1}\tilde{w}_{A2})(\tilde{w}_{B1}+w_{B2})\right). \tag{20}$$

The modified generator of this current in the basis $\{s_0,s_{1A},s_{1B},s_2,s_{3A},s_{3B},s_4,s_{5A},s_{5B}\}$ is given by the following square matrix

$$\hat{H} = \begin{pmatrix} \lambda_{even} & -e^{\mu}\tilde{w}_{A1} & -\tilde{w}_{B1} & 0 & 0 & 0 & 0 & -e^{-\mu}w_{A2} & -w_{B2} \\ -e^{-\mu}w_{A1} & \lambda_A & 0 & -e^{\mu}\tilde{w}_{A2} & 0 & 0 & 0 & 0 & 0 \\ w_{B1} & 0 & \lambda_B & \tilde{w}_{B2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -e^{-\mu}w_{A2} & -w_{B2} & \lambda_{even} & -e^{\mu}\tilde{w}_{A1} & -\tilde{w}_{B1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{-\mu}w_{A1} & \lambda_A & 0 & -e^{\mu}\tilde{w}_{A2} & 0 & 0 \\ 0 & 0 & 0 & -w_{B1} & 0 & \lambda_B & -\tilde{w}_{B2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -e^{-\mu}w_{A2} & -w_{B2} & \lambda_{even} & -e^{\mu}\tilde{w}_{A1} & -\tilde{w}_{B1} \\ -e^{\mu}\tilde{w}_{A2} & 0 & 0 & 0 & 0 & 0 & -e^{-\mu}w_{A1} & \lambda_A & 0 \\ -\tilde{w}_{B2} & 0 & 0 & 0 & 0 & 0 & -w_{B1} & 0 & \lambda_B \end{pmatrix} \tag{21}$$

in which we have defined

$$\begin{aligned} \lambda_{even} &= w_{A1}+w_{B1}+\tilde{w}_{A2}+\tilde{w}_{B2} \\ \lambda_A &= \tilde{w}_{A1}+w_{A2} \\ \lambda_B &= \tilde{w}_{B1}+w_{B2} \end{aligned} \tag{22}$$

We have investigated the characteristic polynomial of (21) and found that it is not symmetric in the sense that it does not satisfy (12) with any real E. On the other hand, it turns out that this characteristic polynomial can be written as a product of a symmetric polynomial in x of order three and an asymmetric polynomial in x of order six. This means that at least three eigenvalues of the modified generator (21) are symmetric. Numerical

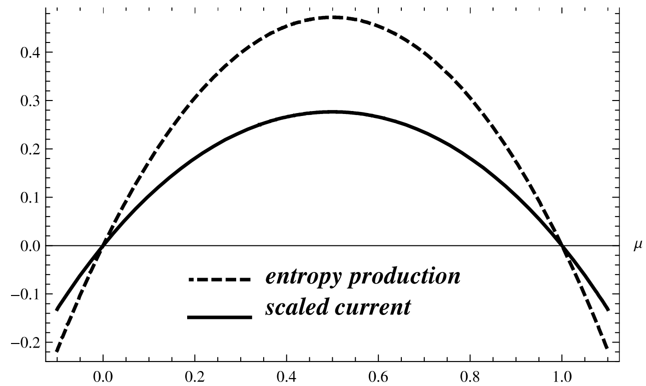

Figure 2. Plot of $e_J\left(E_A\mu\right)$ (filled line) versus $e_S\left(\mu\right)$ (dashed line) for $w_{A1}=4, \tilde{w}_{A1}=1, w_{A2}=3, \tilde{w}_{A2}=2, w_{B1}=3, \tilde{w}_{B1}=1$ and $\tilde{w}_{B2}=1$. As can be seen they have different functionalities.

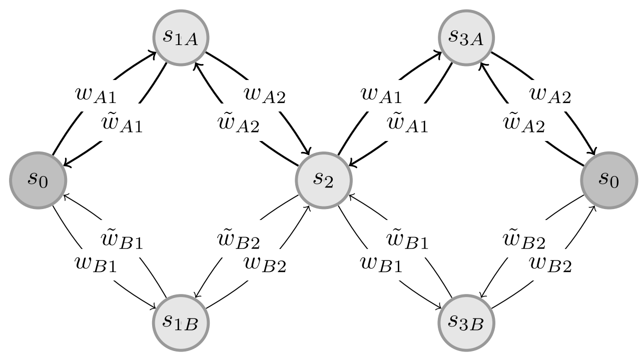

Figure 3. A simple network with GCEM-like symmetry (see inside the text).

investigations show that the minimum eigenvalue of (21) is symmetric, in the sense of (10) with the following conjugate field

$$E_A = \frac{1}{2}\ln\left(\frac{w_{A1}w_{A2}}{\tilde{w}_{A1}\tilde{w}_{A2}}\right) \quad (23)$$

and also vanishes at $\mu=0$ and $\mu=E_A$. This minimum eigenvalue has a very complicated expression; hence it is not presented here. Moreover, we emphasis that the functional form of the minimum eigenvalue for the current $e_J\left(\mu\right)$ differs from that of the entropy production $e_S\left(\mu\right)$.

In Figure 2 we have plotted $e_J\left(E_A\mu\right)$ versus $e_S\left(\mu\right)$ to show that these minimum eigenvalues do not have the same functional form even after rescaling them, hence we call J_A a non-entropic current.

We should also note that the average current (20) vanishes when $w_{A1}w_{A2}=\tilde{w}_{A1}\tilde{w}_{A2}$. The external field (23) also vanishes when this constraint is fulfilled; however, one cannot find a similarity transformation of the form (9) with a real conjugate field E. More precisely, this means that the system will not be in equilibrium when the external field vanishes [6]. Because of the symmetry of our network, we could have considered an equivalent non-entropic time-integrated current $\mathcal{J}_B$ through the following links

$$s_0 \to s_{1B} \to s_2 \to s_{3B} \to s_4 \to s_{5B} \to s_0.$$

It can be shown that the large deviation function of this current has exactly the same symmetries with the conjugate field

$$E_B = \frac{1}{2}\ln\left(\frac{w_{B1}w_{B2}}{\tilde{w}_{B1}\tilde{w}_{B2}}\right). \quad (24)$$

We can also calculate the average entropy production in the steady-state as a function of J_A and J_B as follows [5, 14]

$$\dot{S} = J_A E_A + J_B E_B. \quad (25)$$

Note that the large deviation function associated with the joint probability distribution function of currents J_A and J_B defined by

$$\lim_{t\to\infty} P\left(J_A, J_B, t\right) \sim e^{-t\hat{e}\left(J_A, J_B\right)}. \quad (26)$$

has the GCEM symmetry with respect to the conjugate fields E_A and E_B i.e.

$$\hat{e}\left(-J_A,-J_B\right)-\hat{e}\left(J_A,J_B\right)=E_A\, J_A + E_B\, J_B. \quad (27)$$

In the next section we will explain the origin of this symmetry which turns to be the time-reversal. Although the new symmetry introduced in this paper is similar to the GCEM-like symmetry, in the sense that one should properly group the stochastic trajectories, it has a different nature. In order to see the differences and similarities let us first consider a simpler network of states, similar to the one introduced in the previous section, as it is shown in Figure 3 with a time-integrated current through the following states

$s_0 \to s_{1A} \to s_2 \to s_{3A} \to s_0$

This path is shown in bold in Figure 3. The characteristic polynomial of this current is symmetric with the same conjugate field given in (23). This means that all of the eigenvalues of the modified generator for this current are symmetric including the minimum one. There are two different cycles with increment 0 in this network and their weights are given by

$$W_0^1 \equiv W_{s_0\to s_{1B}\to s_2\to s_{3B}\to s_0} = w_{B1}^2 w_{B2}^2,$$

$$W_0^2 \equiv W_{s_0\to s_{3B}\to s_2\to s_{1B}\to s_0} = \tilde{w}_{B1}^2 \tilde{w}_{B2}^2.$$

Note that these two cycles are time-reversed of each other. There are also four different cycles with increment 2 (and their time-reversed with increment -2) whose weights are given by

$$W_2^1 \equiv W_{s_0\to s_{1A}\to s_2\to s_{1B}\to s_0} = w_{A1} w_{A2} \tilde{w}_{B1} \tilde{w}_{B2},$$

$$W_2^2 \equiv W_{s_2\to s_{3A}\to s_0\to s_{3B}\to s_2} = w_{A1} w_{A2} \tilde{w}_{B1} w_{B2},$$

$$W_2^3 \equiv W_{s_0\to s_{1A}\to s_2\to s_{3B}\to s_0} = w_{A1} w_{A2} w_{B1} w_{B2},$$

$$W_2^4 \equiv W_{s_0\to s_{1B}\to s_2\to s_{3A}\to s_0} = w_{A1} w_{A2} w_{B1} w_{B2}.$$

Finally, there is a cycle with increment 4 (and its time-reversed with increment -4 with the weight

$$W_4^1 \equiv W_{s_0\to s_{1A}\to s_2\to s_{3A}\to s_0} = w_{A1}^2 w_{A2}^2.$$

Denoting the weight of a time-reversal cycle by $\bar{W}$ one can immediately see that

$$\frac{W_4^1}{\bar{W}_4^1} = e^{KE}$$

for $K = 4$ and E given in (23). By properly grouping the cycles with equal increments we also find

$$\frac{W_2^1 + W_2^2 + W_2^3 + W_2^4}{\bar{W}_2^1 + \bar{W}_2^2 + \bar{W}_2^3 + \bar{W}_2^4} = e^{KE}$$

for $K = 2$ and

$$\frac{W_1^0 + W_2^0}{\bar{W}_1^0 + \bar{W}_2^0} = e^{KE}$$

For $K = 0$. These relations are the origin of the symmetry of the characteristic polynomial for the current defined above. Grouping of different cycles with equal increments indicates that this symmetry is a GCEM-like symmetry.

Let us now consider the network of states given in Figure 1. Comparing this network of states with the one given in Figure 2, we see that only the dimensionality of the network is changed while its structure is almost preserved. As we will see this will result in a completely different symmetry for the large deviation function of the current. In this network there are two cycles with increment $K = 0$ with the following weights

$$W_0^1 \equiv W_{s_0\to s_{1B}\to s_2\to s_{3B}\to s_4\to s_{5B}\to s_0} = w_{B1}^3 w_{B2}^3,$$

$$W_0^2 \equiv W_{s_0\to s_{5B}\to s_4\to s_{3B}\to s_2\to s_{1B}\to s_0} = \tilde{w}_{B1}^3 \tilde{w}_{B2}^3.$$

As in the previous network, these two paths are time-reversal of each other. For the following cycles

$$W_2^1 \equiv W_{s_0\to s_{1A}\to s_2\to s_{3B}\to s_4\to s_{5B}\to s_0} = w_{A1} w_{A2} w_{B1}^2 w_{B2}^2,$$

$$W_2^2 \equiv W_{s_0\to s_{1B}\to s_2\to s_{3A}\to s_4\to s_{5B}\to s_0} = w_{A1} w_{A2} w_{B1}^2 w_{B2}^2,$$

$$W_2^3 \equiv W_{s_0\to s_{1B}\to s_2\to s_{3B}\to s_4\to s_{5A}\to s_0} = w_{A1} w_{A2} w_{B1}^2 w_{B2}^2,$$

$$W_2^4 \equiv W_{s_0\to s_{1A}\to s_2\to s_{1B}\to s_0} = w_{A1} w_{A2} \tilde{w}_{B1} \tilde{w}_{B2},$$

$$W_2^5 \equiv W_{s_2\to s_{3A}\to s_4\to s_{3B}\to s_2} = w_{A1} w_{A2} \tilde{w}_{B1} \tilde{w}_{B2},$$

$$W_2^6 \equiv W_{s_4\to s_{5A}\to s_0\to s_{5B}\to s_4} = w_{A1} w_{A2} \tilde{w}_{B1} \tilde{w}_{B2},$$

we obtain $K = 2$. There are also three cycles with $K = 4$

$$W_4^1 \equiv W_{s_0\to s_{1A}\to s_2\to s_{3A}\to s_4\to s_{5B}\to s_0} = w_{A1}^2 w_{A2}^2 w_{B1} w_{B2},$$

$$W_4^2 \equiv W_{s_0\to s_{1B}\to s_2\to s_{3A}\to s_4\to s_{5A}\to s_0} = w_{A1}^2 w_{A2}^2 w_{B1} w_{B2},$$

$$W_4^3 \equiv W_{s_0\to s_{1A}\to s_2\to s_{3B}\to s_4\to s_{5A}\to s_0} = w_{A1}^2 w_{A2}^2 w_{B1} w_{B2},$$

and a single cycle with $K = 6$

$$W_6^1 \equiv W_{s_0\to s_{1A}\to s_2\to s_{3A}\to s_4\to s_{5A}\to s_0} = w_{A1}^3 w_{A2}^3.$$

It can be easily seen that one cannot group the cycles with equal increments to satisfy (16). In more detail, it is possible to group the cycles with $K = 0$ or $K = 6$ to satisfy (16) with the conjugate field given in (23); however, it is not possible to group the cycles with $K = 2$ or $K = 4$. This results in an asymmetric characteristic polynomial. This is the reason why we emphasize that the nature of the new symmetry introduced in this paper is somewhat different from those studied before. In what follows we show that the origin of the symmetry of the large deviation function for the probability distribution function of J_A is the time-reversal.

Let us first define $s_{\text{even}} \in \{s_0, s_2, s_4\}$, $s_A \in \{s_{1A}, s_{3A}, s_{5A}\}$ and $s_B \in \{s_{1B}, s_{3B}, s_{5B}\}$. Starting with a uniform initial distribution we decompose a stochastic trajectory with M jumps in the time interval $[0, t]$ as a sequence of chronological jumps from s_{even} to another s_{even}. In Figure 4 we have plotted a stochastic trajectory, which without loss of generality can be started from s_{even} at $t_0 = 0$. It can be seen that each trajectory returns to s_{even} after two consecutive jumps. It can be realized that eight different events can occur when the system jumps from s_{even} to another s_{even}. The weights of these events are given by

$$e^{-\lambda_{\text{even}}\Delta t_{\text{even}}} w_{A1} e^{-\lambda_A \Delta t_A} \tilde{w}_{A1},$$

$$e^{-\lambda_{\text{even}}\Delta t_{\text{even}}} \tilde{w}_{A2} e^{-\lambda_A \Delta t_A} w_{A2},$$

$$e^{-\lambda_{\text{even}}\Delta t_{\text{even}}} w_{B1} e^{-\lambda_B \Delta t_B} \tilde{w}_{B1},$$

$$e^{-\lambda_{\text{even}}\Delta t_{\text{even}}} \tilde{w}_{B2} e^{-\lambda_B \Delta t_B} w_{B2},$$

$$e^{-\lambda_{\text{even}}\Delta t_{\text{even}}} w_{A1} e^{-\lambda_A \Delta t_A} w_{A2},$$

$$e^{-\lambda_{\text{even}}\Delta t_{\text{even}}} \tilde{w}_{A2} e^{-\lambda_A \Delta t_A} \tilde{w}_{A1},$$

$$e^{-\lambda_{\text{even}}\Delta t_{\text{even}}} w_{B1} e^{-\lambda_B \Delta t_B} w_{B2},$$

$$e^{-\lambda_{\text{even}}\Delta t_{\text{even}}} \tilde{w}_{B2} e^{-\lambda_B \Delta t_B} \tilde{w}_{B1},$$

in which Δt_{even}, Δt_A and Δt_B are the waiting-times in a given s_{even}, s_A and S_B respectively, before a jump occurs. These weights are the building blocks for the weight of an arbitrary stochastic trajectory in the sense that the weight of a stochastic trajectory can be written as a product of different powers of these weights. Now

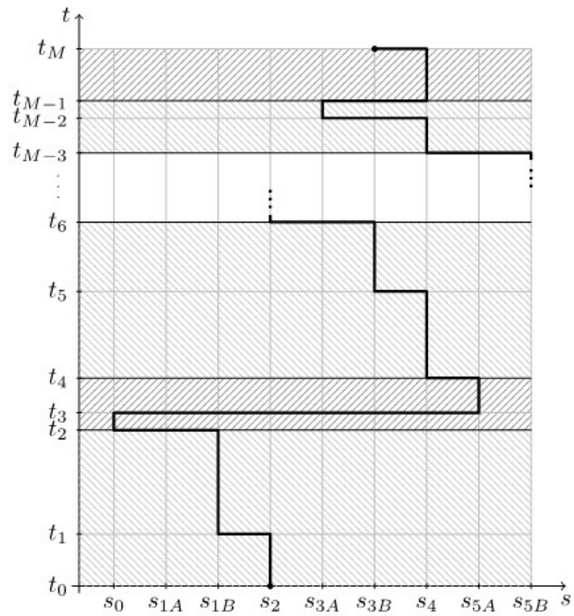

Figure 4. A stochastic trajectory (bold line) can be decomposed into the jumps from s_{even} to another s_{even} shown as different shaded areas. Time points upward and the horizontal axis is the state of the system.

the weight of a stochastic trajectory can be written as

$$\begin{aligned} W\left[\vec{S}_{M,t}\right] = & \left(w_{A1}\tilde{w}_{A1}\right)^{n_{A11}}\left(w_{A2}\tilde{w}_{A2}\right)^{n_{A22}} \\ & \times\left(w_{B1}\tilde{w}_{B1}\right)^{n_{B11}}\left(w_{B2}\tilde{w}_{B2}\right)^{n_{B22}} \\ & \times\left(w_{A1}w_{A2}\right)^{n_{A12}}\left(\tilde{w}_{A1}\tilde{w}_{A2}\right)^{\tilde{n}_{A12}} \\ & \left(w_{B1}w_{B2}\right)^{n_{B12}}\left(\tilde{w}_{B1}\tilde{w}_{B2}\right)^{\tilde{n}_{B12}} \\ & \times\left(w_{B1}w_{B2}\right)^{n_{B12}}\left(\tilde{w}_{B1}\tilde{w}_{B2}\right)^{\tilde{n}_{B12}} \\ & e^{-\lambda_{even}t_{even}-\lambda_A t_A-\lambda_B t_B} \end{aligned}$$

in which the total writing-times in different states are given by

$$t_{even}=\sum\Delta t_{even},\, t_A=\sum\Delta t_A,\, t_B=\sum\Delta t_B$$

Where $t=t_{even}+t_A+t_B$. On the other hand, one should require

$$\begin{aligned} & n_{A11}+n_{A22}+n_{B11}+n_{B22} \\ & \quad +n_{A12}+\tilde{n}_{A12}+n_{B12}+\tilde{n}_{B12}=\frac{M}{2} \end{aligned}$$

in order to fix the total number of jumps M. Using these definitions, we have $\mathcal{J}_A=2\left(n_{A12}-\tilde{n}_{A12}\right)$ and $\mathcal{J}_B=2\left(n_{B12}-\tilde{n}_{B12}\right)$. Let us now consider two different stochastic trajectories and their time-reversals where their only difference is the value of $\mathcal{J}_B$. The first trajectory with $+\mathcal{J}_B$ and the weight

$$\begin{aligned} W\left[\vec{S}_{M,t},\mathcal{J}_A,\mathcal{J}_B\right] = & \left(w_{A1}\tilde{w}_{A1}\right)^{n_{A11}}\left(w_{A2}\tilde{w}_{A2}\right)^{n_{A22}} \\ & \left(w_{B1}\tilde{w}_{B1}\right)^{n_{B11}}\left(w_{B2}\tilde{w}_{B2}\right)^{n_{B22}}\times\left(w_{A1}w_{A2}\right)^{n_{A12}} \\ & \left(\tilde{w}_{A1}\tilde{w}_{A2}\right)^{\tilde{n}_{A12}}\left(w_{B1}w_{B2}\right)^{n_{B12}}\left(\tilde{w}_{B1}\tilde{w}_{B2}\right)^{\tilde{n}_{B12}} \\ & e^{-\lambda_{even}t_{even}-\lambda_A t_A-\lambda_B t_B} \end{aligned}$$

and the second trajectory with $-\mathcal{J}_B$ and the weight

$$\begin{aligned} W\left[\vec{S}_{M,t},\mathcal{J}_A,-\mathcal{J}_B\right] = & \left(w_{A1}\tilde{w}_{A1}\right)^{n_{A11}}\left(w_{A2}\tilde{w}_{A2}\right)^{n_{A22}} \\ & \left(w_{B1}\tilde{w}_{B1}\right)^{n_{B11}}\left(w_{B2}\tilde{w}_{B2}\right)^{n_{B22}}\times\left(w_{A1}w_{A2}\right)^{n_{A12}} \\ & \left(\tilde{w}_{A1}\tilde{w}_{A2}\right)^{\tilde{n}_{A12}}\left(w_{B1}w_{B2}\right)^{\tilde{n}_{B12}}\left(\tilde{w}_{B1}\tilde{w}_{B2}\right)^{n_{B12}} \\ & e^{-\lambda_{even}t_{even}-\lambda_A t_A-\lambda_B t_B}. \end{aligned}$$

After some straightforward calculations it can be shown that the ratio of the sum of the weights of these two trajectories to their time-reversals is given by

$$\begin{aligned} & \frac{W\left[\vec{S}_{M,t},\mathcal{J}_A,\mathcal{J}_B\right]+W\left[\vec{S}_{M,t},\mathcal{J}_A,-\mathcal{J}_B\right]}{W\left[\bar{S}_{M,t},-\mathcal{J}_A,-\mathcal{J}_B\right]+W\left[\bar{S}_{M,t},-\mathcal{J}_A,\mathcal{J}_B\right]} \\ & \quad =\left(\frac{w_{A1}w_{A2}}{\tilde{w}_{A1}\tilde{w}_{A2}}\right)^{n_{A12}-\tilde{n}_{A12}}=e^{E_A\mathcal{J}_A} \end{aligned}$$

in which the conjugate field E_A is given in (23). Multiplying the above expression by a delta function $\delta\left(\mathcal{J}_A-\mathcal{J}\right)$ and summing over all possible trajectories, one recovers the fluctuation theorem given in (15).

6. Concluding remarks

In this paper we have tried to answer the question that has been addressed in recent papers on the symmetries of the large deviation functions other than the GCEM and GCEM-like symmetries. The question is about the existence of a situation where the characteristic polynomial of the modified generator of a non-entropic current is not symmetric but the minimum eigenvalue of the modified generator is symmetric. By introducing a simple network of microscopic transition rates, we have found a non-entropic time-integrated current which has the above-mentioned symmetry. We have investigated all symmetry aspects of the large deviation function for the probability distribution function of this current. We have shown that the origin of this symmetry is time-reversal; however, unlike other GCEM-like symmetries studied in the literature, the cycles with equal increments cannot be grouped, hence in order to recover the fluctuation theorem we have adopted a different approach. It turns out that the existence of this symmetry highly depends on the dimensionality and structure of the network as a given time-integrated current has different symmetries when the network is slightly changed.

Studying of the symmetries of the large deviation functions and the reasons why the large deviation principle can be broken is still an active field [16]. There are still many open questions. For instance it would be very interesting to investigate the necessary conditions on microscopic transition rates which impose the same symmetry studied in this paper in an arbitrary network of states.

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