# Free oscillations of the earth-like planets in the presence of magnetic field

## Y Abedini[1] and S Nasiri[1,2]

1. Department of Physics, Zanjan University, P.O.Box 313, Zanjan, Iran
Email: abedini834@yahoo.com
2. Institute for Advanced Studies in Basic Sciences, Zanjan, Iran
Email: nasiri@iasbs.ac.ir

**Abstract**

we study the free oscillations of a non-rotating earth-like planet in the presence of a force free magnetic field. The model consists of a solid inner core, a liquid outer core and a solid mantle which is spherically symmetric. The lagrangian displacements are decomposed into scaloidal, poloidal and toroidal components using a gauged version of Helmholtz theorem. These components are identified, with $p$-, $g$- and $t$-modes, respectively. The normal modes of the model are determined using a Rayleigh-Ritz variational technique. The consequence of the presence of the solid parts and the magnetic field is the emergence of pure $t$-oscillations. The magnetic field, in addition to exciting $t$-modes, couples the everpresent $p$-and $g$-modes together. As an application of the model, the real seismic data of the earth is used to calculate eigenvalues and eigenvectors for different modes.

## 1. Introduction

Free oscillations of the earth received much attention by the pioneering work of Beniove [1, 2]. He observed an oscillation period of 57 minutes by investigating the seismic data obtained by Kamchatka earthquake and interpreted it as the free oscillation of the earth. Since then many aspects of the problem have been investigated by different authors. Alterman et al. [3] studied the same problem and obtained the numerical values for the periods of free oscillations by means of Runge-Kutta procedure. They classified the normal modes into spheroidal and toroidal components using the Bullen model [4] as a model of the internal structure of the earth. Dynamical equations of the earth obtained by Phinney et al. [5] and Smith [6] were solved by Wiggins [7] using the Rayleigh-Ritz variational procedure and minimizing the action integral for a spherical, non rotating and perfectly elastic model. The theory of free oscillations of the earth is extensively studied by Gilbert et al. [8], Teisseyre [1], Gubbins [9], Sobouti [10] and others, and an enormous body of literature and numerical data are produced. Here, we study the free oscillations of the non rotating earth-like planets by assuming a three layer model consisting of a solid inner core, a liquid outer core and a solid mantle. The mathematical technique, used here to study and classify the possible motions of the model, was first proposed by Cowling

[11] for fluids. In this method, the free oscillations of heavenly fluid bodies are classified into acoustic ($p$-) and gravity ($g$-) modes. The $p$-modes are generated mainly by pressure fluctuations while the $g$-modes are mainly due to density fluctuations [12]. In a medium consisting of liquid and solid, in addition to the pressure and buoyancy forces in liquid regions, the shear forces are also operative in solid parts. Therefore, a third category of modes, i.e. the toroidal modes emerges. These are pure toroidal motions without coupling with everpresent $p$-and $g$-modes. Such a model for earth with liquid core and solid mantle was considered by Sobouti [10] using the same technique and the possible motions were classified assuming a polytropic stratification. Abedini [13] used the same method to study the free oscillations of the above model by employing the real seismic data of the earth given by Gilbert et. al. and identified some of the observed frequencies of spheroidal and toroidal oscillations with those of the calculated values for $p$-, $g$-and $t$-motions.

Furthermore, the oscillations of magnetized fluids were studied, using the same technique, by Sobouti [14], Nasiri et. al. [15], Hasan et.al. [16] and Nasiri [17]. They showed that in the presence of a magnetic field the toroidal modes are excited as standing hydrodynamic waves. Also, the $p$- and $g$-modes are coupled with

$t$-modes. The coupling is proportional to the strength of the magnetic field.

Here, we investigate the possible motions occurring in the aforementioned model immersed in a nonuniform force free magnetic field. In section 2 the dynamical equations for the model are given. In section 3 the structure of lagrangian displacements occurring in the medium are described. In section 4 the computational procedure and results are discussed.

## 2. Equations

Let us consider an istropic system with a solid inner core and liquid outer core and solid mantle in equilibrium. Assume $\rho(r)$, $p(r)$, $U(r)$, and $\mu(r)$ be the density, pressure, gravitational potential, and the rigidity coefficient, respectively. The medium is pervaded by a force free magnetic field.

Let the system undergo a small perturbation, and a mass element at position **r** be displaced by a small distance, $\xi$(**r**,t). Associated with this displacement are the Eulerian changes $\delta\rho, \delta P, \delta U$, and $\delta B$. It can be shown that the linearized equation of motion is

$$\rho \frac{\partial^2 \xi}{\partial t^2} = w\xi, \tag{1}$$

where

$$w_{ij}\xi_i = \frac{\partial}{\partial x_i}\delta P + \delta\rho \frac{\partial U}{\partial x_i} + \rho \frac{\partial}{\partial x_i}\delta U - $$
$$\frac{1}{4\pi}\varepsilon_{ijk}\varepsilon_{jlm}\left[\frac{\partial}{\partial x_l}(\delta B)_m B_k + \frac{\partial}{\partial x_l}B_m(\delta B)_k\right]$$
$$-\frac{\partial}{\partial x_j}[\mu(\frac{\partial\xi_i}{\partial x_j} + \frac{\partial\xi_j}{\partial x_i} - \frac{2}{3}\frac{\partial\xi_r}{\partial x_r}\delta_{ij})], \tag{2}$$

and

$$\delta\rho = -\rho\frac{\partial\xi_i}{\partial x_i} - \xi_i\frac{\partial\rho}{\partial x_i}, \tag{3}$$

$$\delta P = -\gamma P\frac{\partial\xi_i}{\partial x_i} - \xi_i\frac{\partial P}{\partial x_i}, \tag{4}$$

$$\delta B_i = \varepsilon_{ijk}\varepsilon_{klm}\frac{\partial(\xi_l B_m)}{\partial x_j}, \tag{5}$$

$$\frac{\partial^2(\delta U)}{\partial x_i\partial x_i} = 4\pi G\delta\rho. \tag{6}$$

In the above equations the summation over repeated indices is implied. In eq. (4), $\gamma$ is the ratio of specific heats that can be obtained by assuming an appropriate equation of the state for earth's interior. The equation of the state used here is that of Murnaghan and Birch [4]. It is obtained for an isotropic system by thermodynamical considerations as follows

$$P = \frac{3}{2}K_0[(\frac{\rho}{\rho_0})^{\frac{2}{3}} - (\frac{\rho}{\rho_0})^{\frac{5}{3}}], \tag{7}$$

where $K_0$ and $\rho_0$ are bulk modulus and density on the surface of the earth, respectively. Using eq. (7) and assuming adiabatic processes one may obtain the following expression for $\gamma$ [13]

$$\gamma = \frac{\frac{7}{3}(\frac{\rho}{\rho_0})^{\frac{7}{3}} - \frac{5}{3}(\frac{\rho}{\rho_0})^{\frac{5}{3}}}{(\frac{\rho}{\rho_0})^{\frac{7}{3}} - (\frac{\rho}{\rho_0})^{\frac{5}{3}}}. \tag{8}$$

Equation (8) shows the behavior of $\gamma$ as a function of radius through the density variations. If we multiply eq. (1) by $\xi^*$ and integrate over the medium, we get

$$\omega^2\int dv\rho\xi_i^*\xi_i = \int dv\xi_i^* w_{ij}\xi_i^* =$$

$$\int dv\xi\frac{\partial}{\partial x_i}\delta P + \int dv\delta\rho\xi_i^*\frac{\partial}{\partial x_i}U +$$

$$\int dv\rho\xi_i^*\frac{\partial}{\partial x_i}(\delta U) - \frac{1}{4\pi}\varepsilon_{ijk}\varepsilon_{jlm}\int dv\xi_i^*\frac{\partial}{\partial x_l}(\delta B)_m B_k$$

$$-\frac{1}{4\pi}\varepsilon_{ijk}\varepsilon_{jlm}\int dv\xi_i^*\frac{\partial}{\partial x_l}B_m(\delta B)_k + \frac{\mu}{2}\varepsilon_{ijk}\varepsilon_{jlm}\int dv\frac{\partial\xi_k^*}{\partial x_l}\frac{\partial\xi_m}{\partial x_l}$$

$$+\frac{4}{3}\mu\int dv\frac{\partial\xi_i^*}{\partial x_i}\frac{\partial\xi_j}{\partial x_j}, \tag{9}$$

where $\xi$ is assumed to have an exponential time dependence, $e^{i\omega t}$. Integrating by parts and letting the surface integrals vanish [15], we get

$$W - \omega^2 S = 0, \tag{10}$$

where

$$S = \int dv\rho\xi_i^*\xi_i \rangle 0, \tag{11}$$

and

$$W = W(1) + W(2) + W(3) + W(4) + W(5) + W(6) + W(7), \tag{12}$$

$$W(1) = \int dv\frac{1}{\rho}\frac{dP}{d\rho}\delta\rho^*\delta\rho, \tag{13}$$

$$W(2) = \int dv\alpha P(\frac{\partial\xi_i^*}{\partial x_i})(\frac{\partial\xi_j}{\partial x_j}), \tag{14}$$

where $\alpha = \gamma - \frac{\rho}{P}\frac{dP}{d\rho}$,

$$W(3) = -G\iint dvdv'\delta\rho^*\delta\rho\ |\ r - r'\ |^{-1}, \tag{15}$$

$$W(4) = \frac{1}{4\pi}\int dv\delta B_i^*\delta B_i, \tag{16}$$

$$W(5) = \frac{-1}{4\pi}\int dv\delta B_i\varepsilon_{ijk}\xi_j^* B_k, \tag{17}$$

$$W(6) = \frac{\mu}{2}\int dv\varepsilon_{ijk}\varepsilon_{klm}\int dv(\frac{\partial\xi_j^*}{\partial x_l})(\frac{\partial\xi_i}{\partial x_m}), \tag{18}$$

$$W(7) = \frac{4}{3}\mu\int dv(\frac{\partial\xi_i^*}{\partial x_i})(\frac{\partial\xi_j}{\partial x_j}).\qquad(19)$$

It can be shown that the expressions for W(1)-W(7) are all real and symmetric. For details of calculations see Nasiri et al. [15].

Now, we assume an axisymmetric force free magnetic field consisting of toroidal and poloidal components as follows [18]

$$B = B_0[\frac{n(n+1)}{\eta r}Z_n(\eta r)Y_n(\theta),$$
$$\frac{1}{\eta}(\frac{d}{dr}+\frac{1}{r})Z_n(\eta r)\frac{dY_n(\theta)}{d\theta}, \quad Z_n(\eta r)\frac{dY_n(\theta)}{d\theta}],\qquad(20)$$

where

$$Z_n(x) = (\frac{\pi}{2x})^{1/2}j_{n+1/2}(x),\qquad(21)$$

is a spherical Bessel function and $Y_n(\theta)$ is a spherical harmonic. Assuming appropriate boundary conditions on the magnetic field gives $\eta r$ as a zero of the Bessel function [15, 19]. Hereafter, we use the first order Bessel function, $n=1$, and its first zero, $\eta r = 4.493409$.

The force free nature of the field would keep the equilibrium configuration of the system spherically symmetric.

## 3. The structure of lagrangian displacements

Using Sobouti's modified form of Helmholtz theorem [20] we decompose a lagrangian displacement as follows

$$\vec{\zeta} = \vec{\zeta}_p + \vec{\zeta}_g + \vec{\zeta}_t.\qquad(22)$$

Equation (22) will be useful in classifying the modes and its various components can be expressed by

$$\vec{\zeta}_p = -\vec{\nabla}x_p,\qquad(23)$$

$$\vec{\zeta}_g = \frac{1}{\rho}\vec{\nabla}\times\vec{\nabla}\times(\mathbf{r}x_g),\qquad(24)$$

$$\vec{\zeta}_t = \frac{1}{\rho}\vec{\nabla}\times\vec{\nabla}\times\vec{\nabla}\times(\mathbf{r}x_t),\qquad(25)$$

where $\mathbf{r}$ is a unit vector in radial direction. Using eqs. (23) - (25) it can be shown that $\vec{\zeta}_P, \vec{\zeta}_g$ and $\vec{\zeta}_t$ are orthogonal to each other [20]. Let the set $\{\vec{\zeta}_P, \vec{\zeta}_g, \vec{\zeta}_t\}$ constitutes a complete set of basis vectors in the Hilbert space of the displacements, H, in which the inner product is

$$\int dv\rho\vec{\zeta}_r^*.\vec{\zeta}_s = finite, \quad r,s = p,g \text{ or } t.\qquad(26)$$

Hence, any eigensolution of eq. (10) can be expanded in terms of the basis set as

$$\xi_r = \sum_s \zeta_s Z_{sr,} \quad r,s = p,g \text{ or } t,\qquad(27)$$

where $Z_{rs}$ are constants of expansion and will be considered as variational parameters. Substituting eq.

(27) in eq. (10) and using the Rayleigh–Ritz variational technique to minimize the eigenvalues one gets the following matrix equation [12]

$$WZ = SZE,\qquad(28)$$

where E and Z, are the matrices of eigenvalues and expansion coefficients, respectively. The elements of S and W are obtained by using eqs. (23)-(25) in eqs. (11)-(19) and are given in appendices A, B, C and D.

It is convenient to consider each term of the W matrix, separately. As the $p$ displacements are solely responsible for changes in $\delta\rho$, by eq. (13) the block structure of W(1) is as follows

$$W(1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & W_{pp}(1) & 0 \\ 0 & 0 & 0 \end{bmatrix}.\qquad(29)$$

Since in eq. (14), $\text{div}\,\xi_t = 0$, the W(2) matrix has no contribution from $t$ motions. Its explicit block form is

$$W(2) = \begin{bmatrix} W_{gg}(2) & W_{gp}(2) & 0 \\ W_{pg}(2) & W_{pp}(2) & 0 \\ 0 & 0 & 0 \end{bmatrix}.\qquad(30)$$

It is easily seen from eq. (15) that the block form of W(3) is as follows

$$W(3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & W_{pp}(3) & 0 \\ 0 & 0 & 0 \end{bmatrix}.\qquad(31)$$

W(4) and W(5) are the magnetic terms. Using eqs. (16) and (17) one can obtain the following expressions for their block structures

$$W(4) = \begin{bmatrix} W_{gg}(4) & W_{gp}(4) & W_{gt}(4) \\ W_{pg}(4) & W_{pp}(4) & W_{pt}(4) \\ W_{tg}(4) & W_{tp}(4) & W_{tt}(4) \end{bmatrix},\qquad(32)$$

and

$$W(5) = \begin{bmatrix} W_{gg}(5) & W_{gp}(5) & W_{gt}(5) \\ W_{pg}(5) & W_{pp}(5) & W_{pt}(5) \\ W_{tg}(5) & W_{tp}(5) & W_{tt}(5) \end{bmatrix}.\qquad(33)$$

W(6) and W(7) are due to rigidity. From eqs. (18) and (19) one has

$$W(6) = \begin{bmatrix} W_{gg}(6) & W_{gp}(6) & 0 \\ W_{pg}(6) & W_{pp}(6) & 0 \\ 0 & 0 & 0 \end{bmatrix},\qquad(34)$$

and

$$W(7) = \begin{bmatrix} W_{gg}(7) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & W_{tt}(7) \end{bmatrix}. \qquad (35)$$

Combining eqs. (29)- (35), one can obtain the full structure of the W matrix obtained by

$$W = W(1) + W(2) + W(3) + W(4) + W(5) + W(6) + W(7). \qquad (36)$$

Here, we can draw some conclusions without delving into detailed numerical calculations.

i) For the fluid core in a non- magnetized model , the $p$- and $g$-modes are excited through $W_{pp}(1)$, $W_{pp}(2)$, $W_{gg}(2)$, $W_{pp}(3)$, and are coupled with each other through $W_{pg}(2) = W_{gp}(2)$.

ii) The presence of the magnetic field not only excites the $t$-modes through $W_{tt}(4)$ and $W_{tt}(5)$, but also couples the same modes with the $p$- and $g$-modes through $W_{gt}(4) = W_{tg}(4)$, $W_{gt}(5) = W_{tg}(5)$, $W_{pt}(4) = W_{tp}(4)$, and $W_{pt}(5) = W_{tp}(5)$. The coupling is proportional to magnetic energy and, therefore, will be weak due to the weakness of the magnetic field. As a result, the $g$- and $p$-motions will contain small toroidal components in their displacement vectors and vice versa.

iii) The presence of solid parts excite pure toroidal motions only through $W_{tt}(7)$ which is linearly proportional to the rigidity coefficient. Note that W(6) does not contribute to pure toroidal motions.

The orthogonality of $\zeta_g$, $\zeta_p$ and $\zeta_t$ makes the **S** matrix block diagonal

$$S = \begin{bmatrix} S_{gg} & 0 & 0 \\ 0 & S_{pp} & 0 \\ 0 & 0 & S_{tt} \end{bmatrix}. \qquad (37)$$

The Z matrix has no vanishing blocks, that is

$$Z = \begin{bmatrix} Z_{gg} & Z_{gp} & Z_{gt} \\ Z_{pg} & Z_{pp} & Z_{pt} \\ Z_{tg} & Z_{tp} & Z_{tt} \end{bmatrix}. \qquad (38)$$

The E matrix by definition is diagonal

$$E = \begin{bmatrix} E_{gg} & 0 & 0 \\ 0 & E_{pp} & 0 \\ 0 & 0 & E_{tt} \end{bmatrix} \equiv \begin{bmatrix} \omega_{gg}^2 & 0 & 0 \\ 0 & \omega_{pp}^2 & 0 \\ 0 & 0 & \omega_{tt}^2 \end{bmatrix}. \qquad (39)$$

## 4. Computational procedure and results
Let us define the following dimensionless quantities to resolve the question of units.

$$\mathbf{S} = \rho_c R^5 \overline{S}, \qquad (40)$$

$$\mathbf{W}_{123} = \mathbf{W}(1) + \mathbf{W}(2) + \mathbf{W}(3) = P_c R^3 \overline{\mathbf{W}}_{123}, \qquad (41)$$

$$\mathbf{W}_{mag} = \mathbf{W}(4) + \mathbf{W}(5) = (B_0^2 R^3 / 8\pi)\overline{\mathbf{W}}_{123} \qquad (42)$$

$$\mathbf{W}_{shear} = \mathbf{W}(6) + \mathbf{W}(7) = \beta_c P_c R^3 \overline{\mathbf{W}}_{shear}, \qquad (43)$$

where $\rho_c$, $p_c$, R, $B_o$ and $\beta_c$ are central density, central pressure, physical radius, amplitude of the magnetic field and the ratio of the central values of the rigidity and the pressure, respectively . The barred expressions are dimensionless integrals. Equation (10) now becomes

$$\varpi^2 = \omega^2 / \omega_j^2 = (\overline{\mathbf{W}}_{123} + \lambda \overline{\mathbf{W}}_{maq} + \beta_c \overline{\mathbf{W}}_{shear}) / \overline{S}, \qquad (44)$$

$$\omega_j^2 = \frac{p_c}{\rho_c R^2}, \qquad (45)$$

$$\lambda = \frac{B_0^2}{8\pi\rho_c}, \qquad (46)$$

where $\omega_j$ is the Jeans frequency and is used as the unit of $\omega$ and $\lambda$ is the ratio of magnetic and internal energy densities at the center.

Equation (28) is cast in matrix form and solved by standard algorithms of matrix digonalization for different values of $\lambda$ and rigidity. The results are obtained for real structure of the earth using the data given by Gilbert et al [8, 21]. In Table 1 the eigenvalues of $p$ modes, $\omega_p^2$, for the whole earth (using the seismic data for $\mu$ as a function of radius) and also for its outer liquid core ( $\mu = 0$ ) are given for $l = 2$, $\lambda = 0$ and $0.1$. It is seen that in the presence of magnetic field the mode characteristics as well as the corresponding eigenvalues do not undergo a significant change. The situation is shown schematically in figures 1 and 2. In these figures $\omega_p^2$ for first four modes are plotted versus radial mode numbers. The eigenvalues for the fifth modes, calculated in Tables 1-3, do not have enough accuracy in Rayleigh-Ritz variational technique and are ignored in figures. 1-5. The increasing nature of $\omega_p^2$ with the mode number is not considerably affected by magnetic field in both cases.

Table 2 is the same as Table 1, for $\omega_g^2$. In contrast to $p$-modes, the $g$-modes behave differently in the presence of the magnetic field. The decreasing nature of $\omega_g^2$ is not preserved and increases for higher mode orders for the whole earth. See the 2nd and 3rd columns in Table 2. This has origin in the fragile nature of the $g$-modes compared with the robust nature of the $p$-modes as it is well known. In the outer liquid core, where the unstable convective motions take place and $\omega_g^2$ is negative, the effect of magnetic field becomes more apparent. This is shown in figure 3. In the presence of magnetic field, $\omega_g^2$ tends to increase with the mode

**Table 1.** The p-eigenvalues, $\omega_p^2$ , in unit of $\omega_j^2$ , for the whole earth (left) and the liquid outer core (right) for the first five radial mode numbers, assuming l=2, $\lambda = 0$ and $\lambda = 0,1$. The magnetic field does not change the p-eigenvalues significantly, both for the whole earth and the liquid core.
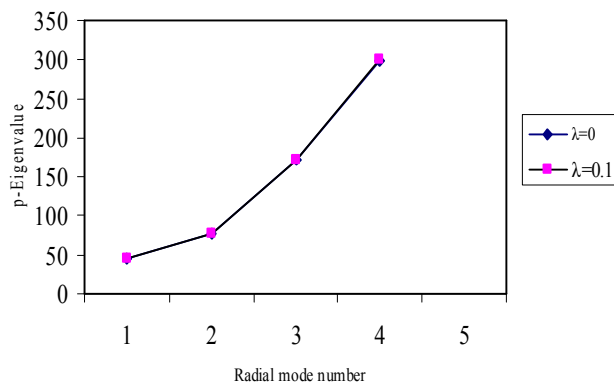
| | The whole earth | | | The liquid core | |
|---|---|---|---|---|---|
| n | $\lambda = 0$ | $\lambda = 0.1$ | n | $\lambda = 0$ | $\lambda = 0.1$ |
| 1 | 0.44169E+2 | 0.44169E+2 | 1 | 0.53918E+2 | 0.53919E+2 |
| 2 | 0.77311E+2 | 0.77319E+2 | 2 | 0.34949E+3 | 0.34956E+3 |
| 3 | 0.17198E+3 | 0.17213E+3 | 3 | 0.80195E+3 | 0.80213E+3 |
| 4 | 0.29905E+3 | 0.29975E+3 | 4 | 0.11175E+4 | 0.11187E+4 |
| 5 | 0.93545E+4 | 0.93634E+4 | 5 | 0.62282E+4 | 0.62291E+4 |

**Table 2.** The same as Table 1 for g-eigenvalues, $\omega_g^2$ . In contrast to p –modes, the situation is different for g –modes . The decreasing nature of g–eigenvalues with mode orders changes for the whole earth and a part of the unstable modes become stable for the liquid core, in the presence of a magnetic field.

| | The whole earth | | | The liquid core | |
|---|---|---|---|---|---|
| n | $\lambda = 0$ | $\lambda = 0.1$ | n | $\lambda = 0$ | $\lambda = 0.1$ |
| 1 | 0.39961E+2 | 0.39992E+2 | 1 | -0.39795E+0 | -0.39166E+0 |
| 2 | 0.51233E+1 | 0.52920E+1 | 2 | -0.12383E+0 | -0.10946E+0 |
| 3 | 0.23145E+1 | 0.24552E+1 | 3 | -0.61086E-1 | -0.20764E-1 |
| 4 | 0.61791E+0 | 0.80207E+0 | 4 | -0.27832E-1 | +0.10417E-1 |
| 5 | 0.12136E+0 | 0.23144E+1 | 5 | -0.55313E-2 | +0.24243E+1 |

**Table 3.** The same as Table 1 for t-eigenvalues, $\omega_t^2$ . The nature of t –modes is preserved , but, a relative increase in eigenvalues is occurred in the presence of magnetic field for the whole earth. For the liquid core, the magnetic field removes the degeneracy of t – modes introducing hydromagnetic waves with relatively long period compared to those produced by the solid part of the earth.

| | The whole earth | | | The liquid core | |
|---|---|---|---|---|---|
| n | $\lambda = 0$ | $\lambda = 0.1$ | n | $\lambda = 0$ | $\lambda = 0.1$ |
| 1 | 0.73832E+1 | 0.76824E+1 | 1 | 0.0 | 0.24771E-1 |
| 2 | 0.13971E+2 | 0.14444E+2 | 2 | 0.0 | 0.13133E+0 |
| 3 | 0.32394E+2 | 0.36684E+2 | 3 | 0.0 | 0.33012E+0 |
| 4 | 0.68158E+2 | 0.89672E+2 | 4 | 0.0 | 0.40395E+0 |
| 5 | 0.24360E+3 | 0.26842E+3 | 5 | 0.0 | 0.80041E+0 |



**Figure 1**. The p-eigenvalues for the whole earth versus radial mode number, assuming 1=2, $\lambda = 0$ and $\lambda = 0.1$. It is seen that the effects of the magnetic filed is not significant on the p-eigenvalues.
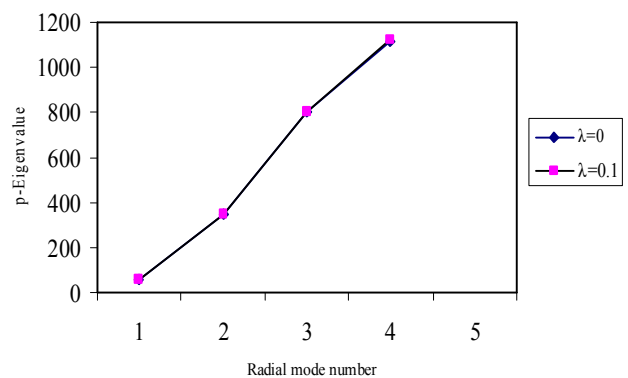
**Figure 2.** The p-eigenvalues for the liquid outer core versus radial mode number, assuming 1=2, $\lambda = 0$ and $\lambda = 0.1$. Again, the situation is not considerably altered by the magnetic field.
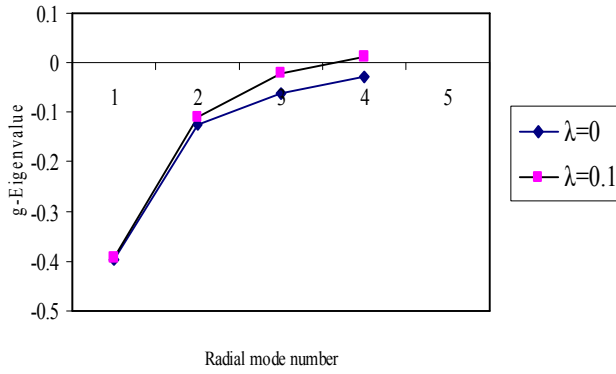
**Figure 3.** The g-eigenvalues for the liquid outer core versus radial mode numbers, assuming 1=2, $\lambda$ =0 and $\lambda$ =0.1. The unstable convective modes in the outer core become stable for n>3 in the presence of the magnetic field.
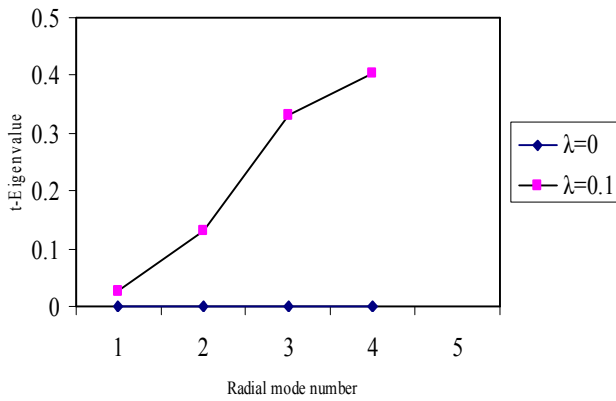


**Figure 5**. The t-eigenvalues for the liquid outer core versus radial mode number, assuming 1=2, $\lambda$ =0 and $\lambda$ =0.1. The magnetic field exits the degenerate t-modes and gives the long period hydromagnetic waves.

order and becomes positive for $n > 3$ giving rise to the oscillatory motions. As a result of this stabilizing effect, the convective motions would be slowed down at the rate that depends on the strength of the magnetic field. Therefore, the rate of the energy, transported by the convective motions would be decreased. This, in turn, may decrease the rate of cooling of the earth's solid core even though by a small rate due to the weakness of the earth's magnetic field.

Numerical values for $\omega_t^2$ are given in Table 3. The toroidal modes are excited either by shear forces or by the magnetic ones. For the whole earth, $\omega_t^2$ undergoes a considerable increase, keeping the modal structure, i.e., the increasing nature, invariant in the presence of the magnetic field. As expected, the $t$ -modes would be degenerate for the nonmagnetized liquid core. This is seen in the 5th column of the Table 3. The presence of magnetic field excites these modes giving standing hydromagnetic waves with long periods compared with those of the seismic waves propagated by shear forces
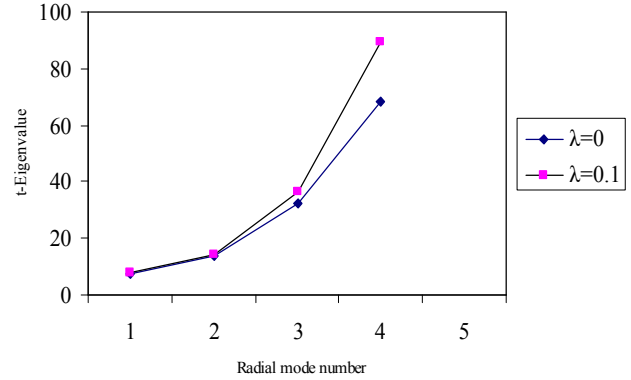


**Figure 4.** The t-eigenvalues for the whole earth versus radial mode number, assuming 1=2, $\lambda$ =0 and $\lambda$ =0.1. The t-eigenvalues undergors a considerable increase in value, but, keeping the modal structure in the presence of the magnetic field.

acting in the solid part of the earth. This is seen from 3$^{rd}$ and 6th columns of Table 3 and the results are plotted in figures 4 and 5.

**Appendix A**
**Matrix elements of S**
The matrix elements of S are as follows

$$S_{pp}^{ij} = \int \rho [\frac{l(l+1)}{r^2} \phi_p^{i'} \phi_p^{j'}] r^2 \, dr,$$

$$S_{gg}^{ij} = \int \frac{1}{\rho}[\frac{l^2(l+1)^2}{r^2} \phi_g^i \phi_q^j + l(l+1)\phi_g'^i \phi_g'^j] r^2 dr,$$

$$S_{tt}^{ij} = \frac{l(l+1)}{2} \int p\phi_t^i \phi_t^j r^2 dr,$$

where the superscripts i and j indicate the order of rows and columns of the matrices.

**Appendix B**
**Matrix elements of W$_{123}$**
Non – zero elements of W$_{123}$ have following elements

$$W_{pp}^{ij}(1) = \int p \frac{dP}{d\rho}[\phi_p''^i + (\frac{2}{r} + \frac{\rho'}{\rho})\phi_p'^i - \frac{l(l+1)}{r^2} \phi_p^i][\phi''^j_p +$$
$$(\frac{2}{r} + \frac{\rho'}{\rho})\phi_p'^j - \frac{l(l+1)}{r^2} \phi_p^j] r^2 dr,$$

$$W_{pp}^{ij}(2) = \int (\gamma - \frac{p}{\rho}\frac{dP}{d\rho})P[\phi''^i_p + \frac{2}{r}\phi_p'^i -$$
$$\frac{l(l+1)}{r^2} \phi_p^i][\phi_p''^j + \frac{2}{r}\phi_p'^j - \frac{l(l+1)}{r^2} \phi_p^j] r^2 dr,$$

$$W_{gp}^{ij}(2) = W_{pg}^{ji}(2) = l(l+1)\int (\gamma - \frac{p}{\rho}\frac{dP}{dp})\frac{P\rho'}{\rho^2}[\phi_P''^i + \frac{2}{r}\phi_P'^i -$$
$$\frac{l(l+1)}{r^2} \phi_p^i]\phi_g^j dr,$$

$$W_{gg}^{ij}(2) = l^2(l+1)^2 \int (\gamma - \frac{p}{\rho}\frac{dP}{d\rho})\frac{P\rho'^4}{r^2\rho^4}\phi_g^i\phi_g^j dr,$$

$$W_{pp}^{ij}(3) = -4\pi G \int \rho Y_p^i Y_p^j dr,$$

where

$$Y_p^i = -\rho\phi_p'^i r + r^l \int_r^R \rho[\frac{l+1}{r}\phi_p'^i - \frac{l(l+1)}{r^2}\phi_p^i]r^{l+1}dr.$$

## Appendix C
### Matrix elements of W_mag terms

The expressions for W(4) and W(5) are more involved than those already encountered. Angular integrals are numerous and complicated. These are denoted by $I_1$- $I_6$. The terms. depending on r are denoted by $a - r$.

    The matrix elements of W(4) and W(5) are calculated as follow

$$W_{pp}^{ij}(4) = \frac{B_o^2}{4\pi}\int [I_1 b_p^i b_p^j + I_2(a_p^i b_p^j + d_p^i c_p^j) +$$
$$I_3(b_p^i a_p^j + c_p^i d_p^j) + 4I_4 c_p^i c_p^j + 4I_5 a_p^i a_p^j]dr,$$

$$W_{gp}^{ij}(4) = W_{pg}^{ji}(4) = \frac{B_o^2}{4\pi}\int [I_1 b_p^i h_g^j + I_2(a_p^i h_g^j + \frac{1}{r}d_p^i m_g^j)$$
$$+ I_2(a_p^i h_g^j + \frac{1}{r}d_p^i m_g^j) + I_6(d_p^i r_g^j + \frac{1}{r^2}f_p^j n_g^i),$$

$$W_{gg}^{ij}(4) = \frac{B_o^2}{4\pi}\int [I_1 m_g^i m_g^j + I_2(a_g^i d_g^j - 2a_g^i b_g^j -$$
$$2b_g^i a_g^j + b_g^i d_g^j + d_g^i b_g^j + d_g^i b_g^j + l_g^i m_g^j + m_g^i l_g^j)$$
$$- 4I_3 a_g^i a_g^i + I_4(a_g^i a_g^j + a_g^i b_g^j + b_g^i a_g^j + b_g^i b_g^j) +$$
$$I_5(l_g^i l_g^j + n_g^i n_g^j + n_g^i r_g^j + r_g^i n_g^j + r_g^i r_g^j)$$
$$+ I_6(a_g^i a_g^j - 2a_g^i d_g^j - 2d_g^i a_g^j + d_g^i b_g^j)]dr,$$

$$W_{pt}^{ij}(4) = W_{tp}^{ji}(4) = -\frac{B_o^2}{4\pi}\int [I_3 e_p^i \phi_t^j + I_6 e_p^i b_t^j]dr,$$

$$W_{tt}^{ij}(4) = \frac{B_o^2}{4\pi}\int [I_1 e_t^i e_t^j + I_2\frac{1}{r}(e_t^i b_t^j + b_t^i e_t^j)$$
$$+ I_5\frac{1}{r^2}b_t^i b_t^j]dr,$$

$$W_{pp}^{ij}(5) = -\frac{B_o^2}{4\pi}\int [I_1\frac{z}{r}b_p^i \phi_p^j - I_2 z^2 \phi_p''^i \phi_p^j + I_6\phi_p^i \phi_p'^j$$
$$+ I_3 zrZC_p^i \phi_p'^j ]dr,$$

$$W_{gp}^{ij}(5) = W_{pg}^{ji}(5) = -\frac{B_o^2}{4\pi}\int [-I_1\frac{z}{\rho r}\phi_p'^i b_g^j + I_2\frac{z^2}{\rho}\phi_p'^i \phi_g''^j +$$
$$I_3 l(l+1)\frac{\eta z}{\rho r}\phi_p^i c_g^j + I_6 l(l+1)\frac{z}{\rho r}\phi_p^i f_g^j]dr,$$

$$W_{gg}^{ij}(5) = -\frac{B_o^2}{4\pi}\int [((2I_2 - I_4)a_g^j - I_4 b_g^i - I_2 d_j^i)\frac{z}{\eta r}\phi_g^j$$
$$(I_2 m_g^i + I_5 l_g^j)\frac{l(l+1)}{\rho r^2}\phi_g^j + I_5(n_g^i + r_g^i)(z+\frac{z'}{r})$$
$$\frac{l(l+1)}{\rho r^2}\phi_g^j - I_2(n_g^j + r_g^i)\frac{2z}{\eta \rho r^2}\phi_p'^j]r^2 dr,$$

$$W_{pt}^{ij}(5) = W_{tp}^{ji}(5) = -\frac{B_o^2}{4\pi}\int [I_2\phi_p^i b_t^j + I_3(z' + \frac{z}{r})\phi_p'^i a_t^j +$$
$$I_4\frac{4z}{r^2}\phi_p^i a_t^j + I_6(z' - \frac{z}{r})\phi_p'^i b_t^j]dr,$$

$$W_{gt}^{ij}(5) = W_{tg}^{ji}(5) =$$
$$- \frac{B_o^2}{4\pi}\int [I_1 e_t^i + I_2 q_t^i)\frac{l(l+1)}{\eta\rho r^2}\phi_g'^j]r^2 dr,$$

$$W_{qt}^{ij}(5) = W_{tq}^{ji}(5) = -\frac{B_o^2}{4\pi}\int [(I_1 e_j^i + I_2 q_t^i)\frac{l(l+1)}{\eta\rho r^2}\phi_q'^j]r^2 dr,$$

$$I_1 = \int_0^\pi \sin^2\theta \, P_l \, P_k' \sin\theta \, d\theta,$$

$$I_2 = \int_0^\pi \sin 2\theta \, P_l P_k' \sin\theta \, d\theta,$$

$$I_3 = \int_0^\pi \sin 2\theta \, P_l' P_k \sin\theta \, d\theta,$$

$$I_4 = \int_0^\pi \cos 2\theta \, P_l' P_k' \sin\theta \, d\theta,$$

$$I_5 = \int_0^\pi \cos^2\theta \, P_l P_k \sin\theta \, d\theta,$$

$$I_6 = \int_0^\pi \sin^2\theta P_l P_k \sin\theta \, d\theta,$$

$$a_p^i = -l(l+1)\frac{i}{\eta r^2}\phi_p''^i + \frac{1}{\eta}(z' + \frac{i}{r})\phi_p'^i,$$

$$b_p^i = -\frac{2}{\eta r^2}\phi_p^i + \frac{1}{\eta}(z' + \frac{z}{r})\phi_p'^i,$$

$$c_p^i = \frac{1}{\eta r}(z' + \frac{z}{r})(\phi_p'^i + z\phi_p^i),$$

$$d_p^i = \frac{1}{\eta}(z' + \frac{z}{r})(\phi_p'^i + r\phi_p''^i),$$

$$e_p^i = -l(l+1)\frac{z}{\eta r^2}\phi_p^i + \frac{1}{\eta}(z' + \frac{z}{r})\phi_p'^i + \frac{z}{\eta}\phi_p''^i,$$

$$f_p^i = \frac{1}{\eta}(z' + \frac{z}{r})\left[2r\phi_p'^i + (2 + \frac{rz'}{z})\phi_p'^i - \frac{l(l+1)}{r}\phi_p^i + \frac{1}{r}(\frac{5}{4} - \eta^2 r^2)\phi_p'^i\right],$$

$$a_t^i = \frac{1}{\eta}\left[\phi_t'^i + (z' - \frac{z}{r})\phi_t^i\right],$$

$$b_t^i = l(l+1)\frac{1}{\eta}(z' + \frac{z}{r})\phi_t^i,$$

$$e_t^i = -\frac{2}{\eta^r}\frac{\theta}{\theta^r}(\frac{z}{r}\phi_t^i),$$

$$a_g^i = \frac{l(l+1)}{\eta pr^3}(z' + \frac{z}{r})\phi_g^i,$$

$$b_g^i = -\frac{2z}{\eta pr^3}\phi_g'^i - \frac{1}{\eta r^3}\left[l(L+1)(z' + \frac{z}{r}\phi_g^i - 2z\phi_g'^i\right],$$

$$d_g^i = \frac{2l(l+1)z}{\eta pr^3}\phi_g'^i,$$

$$f_g^i = \frac{l(l+1)}{\eta r}\left[(z' + \frac{z}{r})\phi_g'^i - \frac{1}{r}(z' + \eta^2 r^2)\phi_g^i\right],$$

$$h_g^i = \frac{l(l+1)}{\eta}\left[(\frac{5}{4} - \eta^2 r^2)\frac{z^2}{r^2} - (z'-1)(z' + \frac{z}{r})\right]\phi_g^i,$$

$$l_g^i = -\frac{l(l+1)}{\eta r}\frac{\partial}{\partial r}\left[\frac{1}{pr}(z' - \frac{z}{r})\phi_g^i\right],$$

$$m_g^i = \frac{2}{\eta r}\frac{\partial}{\partial r}(\frac{z}{\rho r}\phi_g'^i),$$

$$n_g^i = \frac{l(l+1)}{\eta r}\frac{\partial}{\partial r}(\frac{z}{\rho r}\phi_g'^i),$$

$$r_g^i = -\frac{l(l+1)z}{\eta r^2 \rho}\phi_g'^i.$$

## Appendix D
### Matrix elements of $W_{shear}$ terms
The matrix elements of W(6) and W(7) are calculated as follows

$$W_{pp}^{ij}(6) = \frac{4}{3}\int \beta P[\phi_p''^i + \frac{2}{r}\phi_p'^i - \frac{l(l+1)}{r^2}\phi_p^i][\phi_p''^j +$$

$$\frac{2}{r}\phi_p'^j - \frac{l(l+1)}{r^2}\phi_p^j]r^2 dr,$$

$$W_{gp}^{ij}(6) = W_{pg}^{ij}(6) = \frac{4}{3}l(l+1)\int \beta \frac{P\rho'}{\rho^2}[\phi_p''^i + \frac{2}{r}\phi_p'^i$$

$$-\frac{l(l+1)}{r^2}\phi_p^i]\phi_g^j dr,$$

$$W_{qq}^{ij}(7) = \frac{1}{2}l(l+1)\int \beta \frac{P\rho'^2}{\rho^4}\phi_g'^i\phi_g'^j dr,$$

$$W_{tt}^{ij}(7) = \frac{1}{2}l(l+1)\int \beta P[l(l+1)\phi_t^i\phi_t^j + r^2\phi_t'^i\phi_t'^j]dr.$$

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