Energy shift of interacting non-relativistic fermions in noncommutative space

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Abstract
A local interaction in noncommutative space modifies to a non-local one. For an assembly of particles interacting through the contact potential, formalism of the quantum field theory makes it possible to take into account the effect of modification of the potential on the energy of the system. In this paper we calculate the energy shift of an assembly of non-relativistic fermions, interacting through the contact potential in the presence of the two-dimensional noncommutativity.

1. Introduction

In recent years, a good deal of interest is devoted to the physics of noncommutative space -time. In this regard noncommutative quantum electrodynamics is studied in detail and is shown to be renormalizable up to one loop [1], [2]. The same result is obtained for the ϕ^4 theory up to two loops [3], [4]. Now, we know that the idea of the noncommutative space-time leads to non-unitary field theories [5], while noncommutative spaces, are more physically acceptable theories.

On a noncommutative plane, the interaction point shifts by a momentum dependent amount and this leads to the modification of a local interaction to a non-local one [6]. In particular an assembly of the particles interacting through the contact (local) potential will feel a non-local interaction because of the shift of interaction point. This means that two particles can interact with each other while they don't occupy the same place in space. In this paper we calculate the energy of an assembly of non-relativistic fermions interacting through the contact potential in the presence of the two-dimensional noncommutativity.

2. Noncommutative space-time

A noncommutative space-time is characterized by [2], [3], [7]

$$[\hat{x}_\alpha , \hat{x}_\beta] = i\gamma_{\alpha\beta}$$

$$(\gamma_{\alpha\beta} = -\gamma_{\beta\alpha} \ \ and \ \ \alpha,\beta = 0,1,2,3) \quad , \tag{1}$$

then the noncommutativity of space-time for the product of fields is encoded in the Moyal product as

$$\phi_1(x) * \phi_2(x) =$$

$$\exp(\frac{i}{2}\gamma^{\alpha\beta}\partial_{\mu_\alpha}\partial_{\nu_\beta})\phi_1(x+\mu)\phi_2(x+\nu)|_{\mu=\nu=0} \quad , \tag{2}$$

we are interested in noncommutativity of the three-dimensional space i.e. $\gamma_{0\alpha} = 0$,which for the (spatial) exponentials gives

$$e^{i\vec{k}_1 \cdot \vec{x}} * e^{i\vec{k}_2 \cdot \vec{x}} = e^{-i\vec{k}_1 \wedge \vec{k}_2} e^{i(\vec{k}_1 + \vec{k}_2)\cdot \vec{x}} \quad , \tag{3}$$

with

$$\vec{k}_1 \wedge \vec{k}_2 = \frac{1}{2}\gamma_{ij}k_{i1}k_{j2} \qquad (j,i = 1,2,3). \tag{4}$$

3. Non-relativistic fermions in noncommutative space

Interaction Hamiltonian of the instantaneously interacting non-relativistic fermions is [8]

$$H_{\text{int}} = \frac{1}{2} g \int d^3x \, \Psi^+(\vec{x})\Psi^+(\vec{x})\Psi(\vec{x})\Psi(\vec{x}) \quad , \tag{5}$$

where the field operators $\Psi(\vec{x})$ and $\Psi^+(\vec{x})$ are expanded in terms of creation and destruction operators as

$$\Psi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{k},\sigma} \Psi_{\vec{k},\sigma}(\vec{x}) a_{\vec{k},\sigma}$$

$$= \frac{1}{\sqrt{V}} \sum_{\vec{k},\sigma} e^{i\vec{k}\cdot\vec{x}} \zeta_\sigma a_{\vec{k},\sigma} \quad \sigma = 1,2\,(spin\ index) \quad, \qquad (6)$$

$$\Psi^+(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{k},\sigma} \Psi^+_{\vec{k},\sigma}(\vec{x}) a^+_{\vec{k},\sigma}$$

$$= \frac{1}{\sqrt{V}} \sum_{\vec{k},\sigma} e^{-i\vec{k}\cdot\vec{x}} \zeta^+_\sigma a^+_{\vec{k},\sigma} \quad, \qquad (7)$$

with

$$\zeta_1 = \zeta_\downarrow = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad and \quad \zeta_2 = \zeta_\uparrow = \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \qquad (8)$$

In the case of noncommutative space characterized by eq. (1), the interaction Hamiltonian (5) will be replaced by [9]

$$H_{int} = \frac{1}{2} g \int d^3 x\, \Psi^+(\vec{x}) * \Psi^+(\vec{x}) * \Psi(\vec{x}) * \Psi(\vec{x}) . \qquad (9)$$

So after substituting for the field operators from eqs. (6) and(7) in eq. (9) and using eq. (3) one obtains

$$H_{int} = \frac{1}{2} \sum_{\vec{k}_1,\sigma_1} \sum_{\vec{k}_2,\sigma_2} \sum_{\vec{k}_3,\sigma_3} \sum_{\vec{k}_4,\sigma_4} \langle \vec{k}_1,\sigma_1; \vec{k}_2,\sigma_2 | V_{int}(\vec{x}_1 - \vec{x}_2) | \vec{k}_4,\sigma_4; \quad (10)$$

$$\vec{k}_3,\sigma_3 \rangle a^+_{\vec{k}_1,\sigma_1} a^+_{\vec{k}_2,\sigma_2} a_{\vec{k}_4,\sigma_4} a_{\vec{k}_3,\sigma_3}$$

provided that, $\sigma_1 \neq \sigma_2$ which corresponds to the s-wave scattering [8].The interaction matrix reads

$$\frac{1}{V} g \delta_{\sigma_1,\sigma_3} \delta_{\sigma_2,\sigma_4} \delta_{\vec{k}_1+\vec{k}_2=\vec{k}_3+\vec{k}_4}$$

$$e^{-i\vec{k}_1 \wedge \vec{k}_2} e^{-i\vec{k}_3 \wedge \vec{k}_4} e^{i(\vec{k}_1+\vec{k}_2) \wedge (\vec{k}_3+\vec{k}_4)} \qquad (\sigma_1 \neq \sigma_2) \quad (11)$$

so there is momentum conservation for the interacting particles as

$$\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4 \qquad (12)$$

which means that the summations on four wave vectors in eq.(10) must be restricted to the set of three independent vectors. We choose the independent vectors as $\vec{k}, \vec{k}_1\ and\ \vec{k}_3$. The remaining two momenta in eq. (10) can be eliminated via

$$\vec{k}_4 = \vec{k} - \vec{k}_3 \qquad and \qquad \vec{k}_2 = \vec{k} - \vec{k}_1 \quad, \qquad (13)$$

hence the interaction Hamiltonian takes the form

$$H_{int} = \frac{g}{2V} \sum_{\vec{k},\vec{k}_1,\vec{k}_3} \sum_{\sigma_1 \neq \sigma_2} e^{-i\vec{k}_1 \wedge \vec{k}} e^{-i\vec{k}_3 \wedge \vec{k}} a^+_{\vec{k}_1,\sigma_1} a^+_{\vec{k}-\vec{k}_1,\sigma_2} a_{\vec{k}-\vec{k}_3,\sigma_2} a_{\vec{k}_3,\sigma_1} . \qquad (14)$$

First order energy shift of the system (first order in coupling constant) is given by [10]

$$E^{(1)} = \langle F | H_{int} | F \rangle \qquad (15)$$

where $| F \rangle$ stands for the ground state (Fermi vacuum) of the system. In order to get a non vanishing result for the matrix elements of the creation and destruction operators, one must pair them off. The only two possibilities are [10]

$$\begin{cases} \vec{k}_1,\sigma_1 = \vec{k}_3,\sigma_1 \\ (\vec{k}-\vec{k}_1),\sigma_2 = (\vec{k}-\vec{k}_3),\sigma_2 \end{cases} \quad (direct\ pairing) \qquad and$$

$$\begin{cases} (\vec{k}_1,\sigma_1) = (\vec{k}-\vec{k}_3),\sigma_2 \\ (\vec{k}-\vec{k}_1),\sigma_2 = \vec{k}_3,\sigma_1 \end{cases} \quad (exchange\ pairing) \qquad (16)$$

From the condition for the s-wave scattering i.e. $\sigma_1 \neq \sigma_2$, it can be justified that the exchange pairing leads to the vanishing result for the matrix elements of the creation and destruction operators. But for the direct pairing we get

$$\delta_{\vec{k}_1,\vec{k}_3} \langle F | a^+_{\vec{k}_1,\sigma_1} a^+_{\vec{k}-\vec{k}_1,\sigma_2} a_{\vec{k}-\vec{k}_1,\sigma_2} a_{\vec{k}_1,\sigma_1} | F \rangle$$

$$= \delta_{\vec{k}_1,\vec{k}_3} \theta(k_f - | \vec{k} - \vec{k}_1 |) \theta(k_f - k_1) \quad,$$

where the step functions satisfy

$$\theta(x) = 0 \quad for\ x < 0 \quad and \quad \theta(x) = 1 \quad for\ x > 0$$

With this result in hand the first order energy shift reads

$$E^{(1)} = \frac{g}{V} \sum_{\vec{k},\vec{\xi}} e^{2i\vec{k} \wedge \vec{\xi}} \theta(k_f - | \vec{\xi} - \frac{\vec{k}}{2} |) \theta(k_f - | \vec{\xi} + \frac{\vec{k}}{2} |) , \quad (17)$$

where we have redefined the wave vectors as

$$\vec{k}_1 - \vec{k} = (\vec{k}_1 - \frac{\vec{k}}{2}) - \frac{\vec{k}}{2} = \vec{\xi} - \frac{\vec{k}}{2} \quad,$$

$$\vec{k}_1 = (\vec{k}_1 - \frac{\vec{k}}{2}) + \frac{\vec{k}}{2} = \vec{\xi} + \frac{\vec{k}}{2} \quad. \qquad (18)$$

This symmetric form for the first order energy shift has the advantage that now integration on its variables (see below) has a simple geometrical interpretation [10]. The step functions appearing in eq.(17) just confine the magnitudes of the vectors and the angle between them to the maximum and minimum values. The summation is symmetric and hence eq. (17) can be replaced by

$$E^{(1)} = \frac{g}{V} \sum_{\vec{k},\vec{\xi}} \cos(2\vec{k} \wedge \vec{\xi}) \theta(k_f - | \vec{\xi} - \frac{\vec{k}}{2} |) \theta(k_f - | \vec{\xi} + \frac{\vec{k}}{2} |) , \quad (19)$$

for the large amount of volume the summation in eq.(19) can be replaced by integrals as

$$E^{(1)} = \frac{V}{(2\pi)^6} g \int d^3 k \int d^3 \xi\, \theta(k_f - | \vec{\xi} - \frac{\vec{k}}{2} |)$$

$$\theta(k_f - | \vec{\xi} + \frac{\vec{k}}{2} |) \cos(2\vec{k} \wedge \vec{\xi}) . \qquad (20)$$

Now in order to get a concrete result for the energy shift of the system let us assume that the noncommutativity of space is restricted to the two dimensions e.g.

$$\gamma = \gamma_1 = \gamma_{23} = -\gamma_{32} \quad and \quad \gamma_{12} = \gamma_{13} = 0 \quad, \qquad (21)$$

then by substituting in eq.(20) for $\cos(2\vec{k} \wedge \vec{\xi})$ as

$$\cos(2\vec{k} \wedge \vec{\xi}) \approx 1 - \frac{1}{2} \gamma^2 (k_y \xi_z - k_z \xi_y)^2 + O(\gamma^4) , \quad (22)$$

we get the final result for the first order energy shift per

volume as

$$\frac{E^{(1)}}{V} = \frac{1}{36} \cdot \frac{k_f^6}{\pi^4} g - \gamma^2 (3.88 \times 10^{-3}) \frac{k_f^{10}}{\pi^4} g \quad , \qquad (23)$$

where the following list of integrals is used. (Evaluation of the integrals would be straightforward if one takes into the geometrical interpretation of the step functions [10]).

$$\int d^3\xi \, \theta(k_f - |\vec{\xi} - \frac{\vec{k}}{2}|)\theta(k_f - |\vec{\xi} + \frac{\vec{k}}{2}|)$$

$$= \frac{4\pi}{3} k_f^3 (1 - \frac{3}{2}z + \frac{1}{2}z^3)\theta(1-z) \quad (z = k/2k_f)$$

$$\int d^3\xi \, \xi_z^2 \, \theta(k_f - |\vec{\xi} - \frac{\vec{k}}{2}|)\theta(k_f - |\vec{\xi} + \frac{\vec{k}}{2}|)$$

$$= \frac{4\pi}{5} k_f^5 (\frac{1}{3} - \frac{5}{6}z^3 + \frac{1}{2}z^5)\theta(1-z)$$

$$\int d^3\xi \, \xi_y^2 \, \theta(k_f - |\vec{\xi} - \frac{\vec{k}}{2}|)\theta(k_f - |\vec{\xi} + \frac{\vec{k}}{2}|)$$

$$= \frac{4\pi}{5} k_f^5 (\frac{1}{3} - \frac{5}{8}z + \frac{5}{12}z^3 - \frac{1}{8}z^5)\theta(1-z)$$

$$\int d^3\xi \, \xi_i \xi_j = \int d^3k \, k_i k_j = 0 \quad (i \neq j) \quad .$$

The first term of eq.(23) is the first order energy shift in usual commutative space [8], while the second term corresponds to the effect of noncommutativity of space on the energy of system.

4. Conclusion

The effect of noncommutativity of space is to modify the contact potential between the assembly of the non-relativistic fermions, to a non-local interaction. This means that two particles can interact with each other while they don't occupy the same position in space. So it is interesting to calculate the energy shift of the system with modified inter-particle interaction. In this paper we used the field theoretic formalism to incorporate the effect of noncommutativity of space on the energy of the system. Our result implies a reduction in the ground state energy of the system up to the second order in noncommutativity parameter. at low

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