IRANIAN JOURNAL OF PHYSICS RESEARCH

Iranian Journal of Physics Research, Vol. 23, No. 3, 2023
DOI: 10.47176/ijpr.23.3.01769

Electromagnetic medium analogous to Rindler space-time and Poincaré space

Z Hassani*, A Shariati, and M Khorrami
Department of Fundamental Physics, Faculty of Physics, Alzahra University, Tehran, Iran

E-mail: zahrahassani598@gmail.com

(Received 23 October 2023 ; in final form 1 January 2024)

Abstract
By specifying the electric susceptibility tensor ε_{ij} and the magnetic permeability tensor μ_{ij}, we introduce two isotropic but inhomogeneous media which are analogous to the Rindler space-time and the Poincaré half-space. The propagation of electromagnetic waves in these media is investigated.

Keywords: Rindler space-time, Poincaré half-space, electromagnetic waves, transformation optics

1. Introduction

According to general relativity, the space-time is curved, and this curvature leads to various observable effects. Doing experiments in gravitational fields, for example near the horizon of black holes, is not possible. Therefore, it is of great value if one finds a laboratory system which somehow simulates the behavior of a curved space-time. Such systems are said to be analogous to the given gravitational field. If we can find such systems, and if we can make them in a laboratory, then we can do experiments, the results of which indicate information about the gravitational field. For a review of this analogue gravity we refer the reader to [1]. As an example, [2] found a theoretical laboratory analogue of an event horizon of a black hole.

A class of such analog systems consists of ponderable media with non-homogeneous or anisotropic electric susceptibility and magnetic permeability tensors; the medium can also be nonlinear. The system investigated by [3] is such a system, where the relative electric permittivity tensor depends on the fields, $\varepsilon^{\mu}_{\nu} = \varepsilon^{\mu}_{\nu}(\mathrm{E,B})$, and the relative magnetic permeability is a constant isotropic tensor μ.

The analogy of a ponderable medium with a curved space or space-time may be used the other way, *i.e.*, writing Maxwell's equations in a ponderable medium as those defining electromagnetic fields in a curved space or space-time (see [4] for example).

Plebanski has formulated the analogy between an empty curved space-time and the electromagnetic fields of a ponderable media [5].

Transformation optics is a technique that models optical media to space or space-time. Of course, everything is derived from Maxwell's equations in ponderable media. This method creates an equivalence relation between the components of a metric tensor and the permittivity and permeability of a media. Leonhardt and Philbin formulated the refractive index, the electric permittivity, and the magnetic permeability in terms of metrics of a space or space-time [6]. For an almost complete review see [7].

Among the space-times of interest, the Rindler space-time holds particular prominence. It elucidates the observations of observers experiencing constant proper acceleration, each within their frame of reference. Though transformed coordinates reveal the Rindler space-time to be akin to the flat Minkowski variety, accelerated observers therein nonetheless detect noticeable phenomena, such as the materialization of particle pairs.

The Poincaré plane differs in its intrinsic curvature, precluding any change of variables able to recast it as a Minkowski space-time.

However, a relation of conformal equivalence unites the Poincaré and Rindler spaces. Therefore, the propagation of massless particulates shares a common character within these two realms.

In this paper, we investigate Maxwell equations for a ponderable medium analogous to the Rindler space-time, and the Poincaré half-space.

2. The analogy

Source- free Maxwell equations are:

$$\nabla.\mathbf{B} = 0, \tag{1}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{2}$$

$$\nabla.\mathbf{D} = 0, \tag{3}$$

$$\nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}=0. \tag{4}$$

Where ***E*** is the electric field, ***H*** is the magnetic field, ***D*** is the electric displacement, and ***B*** is the magnetic induction vector.

Now consider a space-time with metric

$$ds^2=\sum_{\mu\nu} g_{\mu\nu}dx^{\mu}dx^{\nu}, \tag{5}$$

Where the space-time coordinates are denoted by x^{μ} and $\mu,\nu=0,1,2,3$

As shown by Plebanski in [5], in a *curved empty* space the constitutive relations are given by:

$$\mathbf{D}=\varepsilon_0\varepsilon\mathbf{E}+\frac{\mathbf{w}}{c}\times\mathbf{H}, \tag{6}$$

$$\mathbf{B}=\mu_0\mu\mathbf{H}-\frac{\mathbf{w}}{c}\times\mathbf{E}, \tag{7}$$

where the permittivity ε and the permeability μ are symmetric tensors given by

$$\varepsilon^{ij}=\mu^{ij}=-\frac{\sqrt{-g}}{g_{00}}g^{ij}, \tag{8}$$

and the vector ***w*** has components given by

$$w_i=\frac{g_{0i}}{g_{00}}, \tag{9}$$

The vector ***w*** is the magneto-electric coupling parameter, coupling the magnetic and the electric fields; $g^{\mu\nu}$ are the elements of the inverse matrix the elements of which are $g_{\mu\nu}; i, j=1,2,3$, and *g* is the determinant of the matrix with elements $g_{\mu\nu}$.

In summary, Maxwell's equations in a curved empty space are, in form, the same as Maxwell's equations in Cartesian coordinates of an Euclidean space for a ponderable medium with certain constitutive equations. This enables us to design ponderable media corresponding to non-Euclidean space-times.

Media with

$$\varepsilon^{ij}=\mu^{ij}, \tag{10}$$

are called *impedance-matched* media. The terminology is derived from the definition of the impedance of isotropic ponderable media

$$Z=\sqrt{\frac{\mu\mu_0}{\varepsilon\varepsilon_0}}. \tag{11}$$

3. The Rindler space-time

The Rindler space-time is described as the line element

$$ds^2=-\frac{z^2}{a^2}c^2dt^2+dx^2+dy^2+dz^2, \tag{12}$$

where *c* is the velocity of light in vacuum and *a* is a characteristic length. For later reference, we write

$$\left[g^{R}_{\mu\nu}\right]=\mathrm{diag}\left(-\frac{z^2c^2}{a^2},1,1,1\right), \tag{13}$$

which means a diagonal matrix with specified diagonal elements. This metric describes the Minkowski space-time as observed by a uniformly accelerated observer. The *proper* acceleration of the particle at (*x; y; z*) is

$$\alpha=\frac{c^2}{z}. \tag{14}$$

As *z* approaches 0, the proper acceleration diverges. The *z* = 0 surface is the horizon of the Rindler space-time. For a review of the basic definitions of the Rindler frame see [8, pp. 150-154].

For a light ray, propagating along the *z*-axis we have $ds^2 = 0$, $dx = dy = 0$, and

$$a\frac{dz}{z}=c\,dt \;\Rightarrow\; \ln\frac{z}{z_0}=\pm\frac{c}{a}(t-t_0), \tag{15}$$

$$z(t)=z_0e^{\pm\frac{c}{a}(t-t_0)}. \tag{16}$$

Using the formalism of section 2, the electric permittivity and the magnetic permeability can be obtained as:

$$\varepsilon^{ij}=\mu^{ij}=\frac{a}{z}\delta^{ij}, \tag{17}$$

$$D=\varepsilon_0\frac{a}{z}E, \qquad B=\mu_0\frac{a}{z}H. \tag{18}$$

Following the formulation given by Born and Wolf in [9, p. 11], we have:

$$\nabla^2E-\frac{\mu\varepsilon}{c^2}\frac{\partial^2E}{\partial t^2}+\left(\nabla\ln\mu\right)\times\nabla\times E +\nabla\left(E\cdot\nabla\ln\mu\right)=0, \tag{19}$$

Using (17) we have:

$$\nabla\ln\varepsilon=-\frac{\hat{k}}{z}, \tag{20}$$

$$\nabla\ln\mu=-\frac{\hat{k}}{z} \tag{21}$$

Now following (19) we get

$$\nabla^2E-\frac{a^2}{z^2c^2}\frac{\partial^2E}{\partial t^2}-\frac{1}{z}\hat{k}\times(\nabla\times E)-\nabla\frac{E_z}{z}=0, \tag{22}$$

$$\nabla^2H-\frac{a^2}{z^2c^2}\frac{\partial^2H}{\partial t^2}-\frac{1}{z}\hat{k}\times(\nabla\times H)-\nabla\frac{H_z}{z}=0. \tag{23}$$

For the time dependence $e^{-i\omega t}$ we get:

$$\nabla^2E+\frac{a^2}{z^2c^2}\omega^2E-\frac{1}{z}\hat{k}\times(\nabla\times E)-\nabla\frac{E_z}{z}=0, \tag{24}$$

$$\nabla^2H+\frac{a^2}{z^2c^2}\omega^2H-\frac{1}{z}\hat{k}\times(\nabla\times H)-\nabla\frac{H_z}{z}=0. \tag{25}$$

4 . The Poincaré space

The three-dimensional Poincaré Space is given by the spatial metric

$$ds^2=\frac{a^2}{z^2}\left(dx^2+dy^2+dz^2\right), \tag{26}$$

where *a* is a characteristic length. This space has constant negative scalar curvature $\kappa=-1/a^2$.

Now let us study the space-time with line element

$$ds^2=-c^2dt^2+\frac{a^2}{z^2}\left(dx^2+dy^2+dz^2\right), \tag{27}$$

or the space-time metric

$$\left[g^{P}_{\mu\nu}\right] = \mathrm{diag}\left(-c^2, \frac{a^2}{z^2}, \frac{a^2}{z^2}, \frac{a^2}{z^2}\right). \tag{28}$$

By replacing in (8) we write

$$\varepsilon = \varepsilon_0 \frac{a}{z}, \qquad \mu = \mu_0 \frac{a}{z}. \tag{29}$$

These are the same as (17), from which we conclude that Maxwell equations in this space-time are the same as the Maxwell equations in the Rindler space-time. This is not surprising, since these two space-times are conformally related, that is, we have:

$$g^{R}_{\mu\nu} = \frac{z^2}{a^2} g^{P}_{\mu\nu}. \tag{30}$$

5. Solving the wave equations

We now investigate some solutions to the wave equations (22, 23) with time dependence $e^{-i\omega t}$, which leads to equations (24, 25).

5. 1. Propagation long the Z-axis

Consider the case when a light signal is moving in the z direction with

$$E = E(z)\hat{i}, \qquad H = H(z)\hat{j}. \tag{31}$$

Maxwell's equations now read:

$$\nabla\times E = \hat{j}\frac{\partial E_z}{\partial z}, \qquad \nabla\times H = -\hat{i}\frac{\partial H_z}{\partial z}, \tag{32}$$

from which it follows that

$$\frac{\partial^2 E}{\partial z^2} + \frac{1}{z}\frac{\partial E}{\partial z} + \frac{a^2\omega^2}{c^2 z^2} E = 0. \tag{33}$$

The solution of this equation is

$$\frac{E(z)}{E_0} = c_1 \sin\left(\frac{a\omega}{c}\ln\frac{z}{z_0}\right) + c_2\cos\left(\frac{a\omega}{c}\ln\frac{z}{z_0}\right), \tag{34}$$

or in complex form

$$E(z) = E_0 e^{\pm i\frac{a\omega}{c}\ln(z/z_0)}. \tag{35}$$

The wave is therefore

$$E(z,t) = \mathrm{Re}\, E_0 e^{\pm i\frac{a\omega}{c}\ln(z/z_0) - i\omega t}. \tag{36}$$

The constant-phase equation for this wave is

$$\pm\frac{a\omega}{c}\ln\frac{z}{z_0} = \omega t \;\Rightarrow\; z = z_0 e^{\pm ct/a}, \tag{37}$$

which is the same as (16).

The constant-phase surfaces given by equation (37) align precisely with the geodesic trajectory equation (16) of test particles in Rindler coordinates. This shows light follows the intrinsic curvature of the emulated space-time.

The magnetic field could be easily derived from the Maxwell equation

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{38}$$

and it is

$$B(z,t) = \mathrm{Re}\left\{E_0\frac{ia}{cz}\left[-c_1\cos\left(\frac{a\omega}{c}\ln\frac{z}{z_0}\right) + c_2\sin\left(\frac{a\omega}{c}\ln\frac{z}{z_0}\right)\right]e^{-i\omega t}\right\}, \tag{39}$$

5. 2. Propagation In the (*x; z*) Plane

Consider an electromagnetic field such that

$$\mathbf{E} = \mathrm{Re}\left(e^{-i\omega t}e^{ikx}\mathbf{F}(z)\right). \tag{40}$$

Inserting this in the wave equation (24) we get:

$$0 = \left[\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} + \frac{\beta^2}{u^2} - \alpha^2\right]F_x - \frac{i\alpha}{u}F_z, \tag{41}$$

$$0 = \left[\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} + \frac{\beta^2}{u^2} - \alpha^2\right]F_y, \tag{42}$$

$$0 = \left[\frac{d^2}{du^2} - \frac{1}{u}\frac{d}{du} + \frac{1+\beta^2}{u^2} - \alpha^2\right]F_z, \tag{43}$$

where

$$\beta = a\omega, \qquad \alpha = ak, \qquad u = \frac{z}{a}. \tag{44}$$

The Maxwell equation $\nabla\cdot\mathbf{D} = 0$ reads

$$\nabla\cdot\mathbf{E} - \frac{E_z}{z} = 0, \tag{45}$$

and using (40) we get

$$F_x = \frac{i}{\alpha}\left(\frac{d}{du} - \frac{1}{u}\right)F_z. \tag{46}$$

Solving equation (43) we can get F_z. Equation (42) is the modified Bessel equation with an imaginary index. The solution reads:

$$F_y = fI_{i\beta}(\alpha u) + eK_{i\beta}(\alpha u), \tag{47}$$

where f and e are constants, and, I_ν , K_ν are modified Bessel functions of the first and second type with indices ν. Similarly, equation (43) has the solutions

$$F_z = g\alpha u I_{i\beta}(\alpha u) + h\alpha u K_{i\beta}(\alpha u), \tag{48}$$

where g and h are arbitrary constants. Now, equation (46) yields

$$F_x = gi\alpha u I'_{i\beta}(\alpha u) + hi\alpha u K'_{i\beta}(\alpha u), \tag{49}$$

where a prime means the derivative with respect to the argument.

The boundary condition is that for $z\to\infty$ the fields must be finite, which leads to the solution

$$\mathbf{F} = h\alpha u\left[iK'_{i\beta}(\alpha u)\hat{i} + K_{i\beta}(\alpha u)\hat{k}\right] + eK_{i\beta}(\alpha u)\hat{j}. \tag{50}$$

Now using the Maxwell equation

$$\nabla\times\mathbf{E} = i\omega\mathbf{B} \tag{51}$$

and

$$\mathbf{H} = \frac{z}{\mu_0 a}\mathbf{B} \tag{52}$$

we get

$$\mathbf{H} = e^{ikx}\mathbf{G}(z), \tag{53}$$

and

$$\mu_0 c\mathbf{G} = e\frac{i\alpha u}{\beta}K'_{i\beta}(\alpha u)\hat{\mathbf{i}} - hK_{i\beta}(\alpha u)\hat{\mathbf{j}} + e\frac{\alpha u}{\beta}K_{i\beta}(\alpha u)\hat{\mathbf{k}}. \qquad (54)$$

The time average of the Poynting vector becomes:

$$\mathbf{S} = \frac{1}{2}\varepsilon_0 c\frac{\alpha}{\beta}\left(|e|^2 + |h|^2\right)u\left[K_{i\beta}(\alpha u)\right]^2\hat{\mathbf{i}} + \mathrm{Im}(he^*)\frac{\alpha^2}{\beta^2}u^2K'_{i\beta}(\alpha u)\hat{\mathbf{j}}. \qquad (55)$$

The special case $k = 0$ is treated separately. Here the electric and magnetic fields are directly obtained from (41 - 43).

$$\mathbf{F} = \mathbf{f}^+\exp(i\beta\ln u) + \mathbf{f}^-\exp(-i\beta\ln u), \qquad (56)$$

$$\mu_0 c\mathbf{G} = \hat{\mathbf{k}}\times\left[\mathbf{f}^+\exp(i\beta\ln u) - \mathbf{f}^-\exp(-i\beta\ln u)\right], \qquad (57)$$

where $\mathbf{f}^+$ and $\mathbf{f}^-$ are arbitrary constant vectors subject to the conditions

$$\hat{\mathbf{k}}\cdot\mathbf{f}^\pm = 0. \qquad (58)$$

Now the time average of the Poynting vector becomes:

$$\mathbf{S} = \frac{\varepsilon_0 c}{2}\left(|\mathbf{f}^+|^2 - |\mathbf{f}^-|^2\right)\hat{\mathbf{k}} \qquad (59)$$

5. 3. Polarization in the (*x, z*) plane

We now show that if the polarization is considered to be in the (x, z) plane, then the wave would be of the form investigated in the previous section, that is equation (40). Consider the electric field as

$$\mathbf{E} = E_x\hat{i} + E_z\hat{k}. \qquad (60)$$

Equation (24) now reads

$$0 = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \frac{a^2\omega^2}{z^2}E_x + \left(-\frac{2}{z}\frac{\partial E_z}{\partial x} + \frac{1}{z}\frac{\partial E_x}{\partial z}\right), \qquad (61)$$

$$0 = -\frac{2}{z}\frac{\partial E_z}{\partial y}, \qquad (62)$$

$$0 = \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} + \frac{a^2\omega^2}{z^2}E_z + \left(\frac{E_z}{z^2} - \frac{1}{z}\frac{\partial E_z}{\partial z}\right). \qquad (63)$$

From the second equation, $E_z(y) =$ constant. Also $\frac{\partial^2 E_z}{\partial y^2} = 0$, So the third equation simplifies to

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial z^2} - \frac{1}{z}\frac{\partial E_z}{\partial z} + \frac{(1 + a^2\omega^2)}{z^2}E_z = 0. \qquad (64)$$

This equation could be solved by separating variables:

$$E_z(x,z) = f(z)g(x), \qquad (65)$$

$$\frac{f''(z)}{f(z)} - \frac{1}{z}\frac{f'(z)}{f(z)} + \frac{1 + a^2\omega^2}{z^2} = -\frac{g''(x)}{g(x)} = \pm\alpha^2. \qquad (66)$$

Case $-\alpha^2$ leads to $g(x) = e^{\pm\alpha x}$ which is not acceptable, because it diverges either at $x \to \infty$ or at $x \to -\infty$. The case $+\alpha^2$ leads to

$$g(x) = e^{\pm i\alpha x}, \qquad (67)$$

$$f(z) = K_{i\beta}(\alpha z), \qquad (68)$$

where

$$\beta = a\omega. \qquad (69)$$

Therefore

$$E_z(x,z) = e^{\pm i\alpha x}K_{i\beta}(\alpha z). \qquad (70)$$

Now equation (61) reads

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \frac{a^2\omega^2}{z^2}E_x + \frac{1}{z}\frac{\partial E_x}{\partial z} = \frac{2}{z}\frac{\partial E_z}{\partial x}. \qquad (71)$$

This is an inhomogeneous linear equation for E_x.
The solution to the homogeneous equation could be found by the separation of variables.

$$E_x(x,y,z) = g(x)h(y)f(z), \qquad (72)$$

leading to

$$\frac{g''(x)}{g(x)} + \frac{h''(y)}{h(y)} + \frac{f''(z)}{f(z)} + \frac{1}{z}\frac{f'(z)}{f(z)} + \frac{a^2\omega^2}{z^2} = 0. \qquad (73)$$

Considering fields independent of y we set $\frac{h''(y)}{h(y)} = 0$, and we get:

$$\frac{f''(z)}{f(z)} + \frac{1}{z}\frac{f'(z)}{f(z)} + \frac{\beta^2}{z^2} = -\frac{g''(x)}{g(x)} = \pm\alpha^2. \qquad (74)$$

The case $-\alpha^2$ leads to $g(x) = e^{\pm\alpha x}$ which is not acceptable, because it diverges either at $x \to \infty$ or at $x \to -\infty$. The case $+\alpha^2$ leads to

$$g(x) = e^{\pm i\alpha x}, \qquad (75)$$

$$f(z) = K_{i\beta}(\alpha z), \qquad (76)$$

$$E_x(x,z) = e^{\pm i\alpha x}K_{i\beta}(\alpha z). \qquad (77)$$

Finally, considering a possible phase difference between the two polarizations, we get

$$\mathbf{E} = E_0\mathrm{Re}\left\{e^{\pm i\alpha x}K_{i\beta}(\alpha z)\boldsymbol{\varepsilon}\right\}, \qquad (78)$$

where $\boldsymbol{\varepsilon}$ is a complex polarization vector, in the (x; z)-plane.

6. The Eikonal equation

In a ponderable medium with refractive index n, the Eikonal equation, describing the rays in geometric optics reads [9, p. 119]

$$\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2=n^2 \qquad (79)$$

The medium we are studying is given by the refractive index

$$n=\sqrt{\mu\varepsilon}=\frac{a}{z}. \qquad (80)$$

Let us consider a solution to this equation with

$$S(x,y,z)=\xi x+f(z). \qquad (81)$$

We note that S has the dimensions of length and ξ is dimensionless. The eikonal equation then reads

$$\xi^2+\left(\frac{df}{dz}\right)^2=\frac{a^2}{z^2}, \qquad (82)$$

or

$$\frac{df}{dz}=\pm\sqrt{\frac{a^2}{z^2}-\xi^2}. \qquad (83)$$

The vector ∇S is tangent to the ray. So that the slope of the ray, in the ($x; z$) plane is

$$m=\frac{dz}{dx}=\frac{\hat{\mathbf{k}}\cdot\nabla S}{\hat{\mathbf{i}}\cdot\nabla S}=\frac{1}{\xi}\frac{df}{dz}=\pm\sqrt{\frac{a^2}{\xi^2 z^2}-1}. \qquad (84)$$

This slope diverges at the *horizon* $z = 0$, and vanishes at the *height*

$$z=\frac{a}{\xi}. \qquad (85)$$

This means that a ray moving away from the horizon, will eventually return to the horizon, unless the ray is moving parallel to the z- axis.

7. Conclusion

Overall, this work offers a compelling proof-of-concept for the transformation optics approach to modeling curved space-times using analog systems. This could illuminate new avenues in fields from cosmology to quantum information.

In essence, by directly solving the wave dynamics using this optical analog and consistently recovering both ray trajectories and field behavior expected from the original curved space-time, we've demonstrated this transformation optics framework faithfully replicates key gravitational phenomena. This establishes the validity of studying more exotic cosmological and quantum gravitational effects in tabletop photonics experiments.

Acknowledgements

This work was supported by the Research Council of the Alzahra University.

9. References

1. C Barcelo, S Liberati, and M Visser, *Living Rev.Rel* **14** (2011) 3.
2. U Leonhardt, *Nature*, **415** (2002) 406.
3. V A De Lorenci, R Klippert, and D H Teodoro, *Phys. Rev. D*, **70** (2004) 124035.
4. V A De Lorenci and M A Souza, *Physics Letters B*, **512**, 3 (2001) 417.
5. J Plebanski, *Phys. Rev*. **118** (1960) 1396.
6. U Leonhardt and Th Philbin, "*Geometry and Light: The Science of Invisibility*". Dover publications (2014).
7. M McCall et al., *J. Opt*. **20**, 6 (2018) 063001.
8. A Shariati. "*A Mathematical Approach to Special Relativity*". Academic Press (2023).
9. M Born and E Wolf. "*Principles of Optics*", Cambridge University Press, London (1980).