Iranian Journal of Physics Research, Vol. 23, No. 3, 2023
DOI: 10.47176/ijpr.23.3.21799

The impact of random memory resetting on a non-Markovian random walk

H Vazini Hekmat and F Jafarpour*
Department of Physics, Faculty of Science, Bu-Ali Sina University, Hamedan, Iran

E-mail: f.jafarpour@basu.ac.ir

(Received 9 December 2023 ; in final form 06 February 2024)

Abstract
In this study, we present a discrete-time non-Markov process, referred to as an Elephant Random Walk, implemented on an infinite one-dimensional lattice, with the inclusion of random memory resetting. Upon each random resetting event, the walker completely loses its memory. Through analytical calculations, we determine the moments of displacement in the presence of random resetting. Our findings demonstrate that the process does not attain a steady state. However, the long-time behavior of the moments reveals that, under specific conditions, the displacement distribution follows a Gaussian distribution. By manipulating the resetting mechanism, the transition from diffusive to superdiffusive behavior, or vice versa, can be induced in the process.

Keywords: non-Markovian process, resetting, moments, Gaussian distribution

1. Introduction

If we look a little carefully at the world we live in, resetting can be observed in many occurrences. Formatting a hard disk, emptying a recycle bin, having a car gas tank emptied, returning home, or housecleaning are all events that can be interpreted as resetting. Stochastic resetting, which has recently attracted the attention of researchers in different fields of science including physics, can be defined in different forms. In all these definitions, stochastic resetting results in a re-initialization to a quantity such as random returning to a fixed location which is the most commonly considered definition of stochastic resetting [1-6]. Another form of stochastic resetting is random returning to a fraction of the distance from the origin [7,8], to the farthest distance of an already visited location from the starting point [9], or to one of the previously visited locations in a random walk [10,11].

Finding a non-trivial non-equilibrium stationary state [4,5] and minimizing the mean first-passage time, which is used in a search process [6,12], are the most common problems raised in many stochastic resetting researches. It has been shown that for a normal diffusion in which the probability of finding the walker comes from a Gaussian distribution, the probability distribution changes to a Laplace distribution when the walker resets to a fixed location [4,5]. Tal-Friedman et al. have shown that even resetting to a fraction of the distance from the origin with or without drift leads to a non-equilibrium steady state [7].

The study of diffusion phenomena is so wide that its traces can be observed in almost all fields of statistical physics. They also appear in biological, social, and economic systems [13-19]. For diffusive phenomena that have a Mean Square Displacement (MSD) of the form $\langle x_t^2 \rangle \sim t^{2H}$, while normal diffusion is defined by $H = 1/2$, anomalous diffusion is defined by $H \neq 1/2$. The exponent H is known as the Hurst exponent. In a superdiffusive regime $H > 1/2$ while in a subdiffusive regime $H < 1/2$. An Elephant Random Walk (ERW) is a microscopic non-Markov model defined on an infinite lattice in discrete time, where the memory of the past plays a key role in the evolution of the walker [20]. In this model, the random walker has memory access to the complete history of the random walk. The ERW and its different modifications show that memory of the past can lead to anomalous diffusions [20-27]. In [28,29], it has been shown that the ERW propagator can be non-Gaussian or Gaussian depending on the parameters of the problem in question. Cressoni et al. [21] have shown that if a random walker remembers only the distant past, normal diffusion switches to superdiffusion hence, its propagator deviates from Gaussian.

Let us assume that the memory of an ERW is subjected to random resetting in the sense that its memory might be erased randomly with a certain probability. It is clear that if resetting occurs too often then the walker rarely uses its memory and the process is Markovian. However, it is not a priori clear what happens when the probability of resetting is not too high. In other words, one could inquire whether resetting the memory has any impact on the statistics of the walker's displacement, resulting in a non-trivial non-equilibrium steady state or inducing anomalous diffusion.

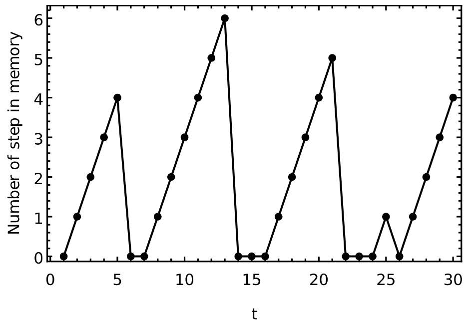

Figure 1. Illustration of the number of steps kept in memory under resetting. At the time step 26, the last reset is occurred; therefore, the 26th step is taken according to the dynamical rule for the first step.

Our exact analytical calculations, besides the numerical results, show that under certain conditions the ERW in the presence of memory resetting, even at extremely low resetting probability, makes the ERW propagator Gaussian for both reformer and traditionalist elephant. By finding the exact expressions for the moments, we show that memory resetting can transform a superdiffusion process into a normal diffusion process.

This paper is organized as follows: In section 2, we briefly review the ERW model. In Section 3, we introduce the ERW model under memory resetting and using the generalized renewal equation, obtain its propagator. In section 4, some recursion relations are solved to obtain the exact expressions for the moments of displacement of the walker. We will also investigate the long-time behaviors of these moments. In section 5, it has been shown, both numerically and analytically, that the ERW under memory resetting obeys Gaussian statistics under certain conditions. The conclusion is presented in section 6.

2. ERW: A quick review

For the sake of completeness, we first restate the ERW defined in [20]. In this model, a random walker performs a discrete-time random walk on a one-dimensional lattice. To take a step at time $t+1$, the elephant looks at all the steps it has taken during the process up to time t. It chooses one step among previously taken steps with a uniform distribution. This step can be either a forward or a backward step. The random walker either accepts the chosen step with probability p, and moves in the direction of the chosen step or does not accept it with probability $1-p$, and moves in the opposite direction. In the first step, the elephant takes one step to the right with probability q or one step to the left with probability $1-q$. Starting from X_0 at time $t_0=0$, the following master equation governs the ERW propagator $P(Y,t+1|X_0,0)$ for $t \geq 1$ [20]:

$$P(Y,t+1|X_0,0) = \frac{1}{2}\left[1-\frac{\alpha}{t}(Y-X_0+1)\right]P(Y+1,t|X_0,0) + \frac{1}{2}\left[1+\frac{\alpha}{t}(Y-X_0-1)\right]P(Y-1,t|X_0,0)\,, \quad (1)$$

where $\alpha = 2p-1$. A negative value of α corresponds to a "reformer" elephant, while a positive value of α corresponds to a "traditionalist" elephant. It is clear that $\alpha=0$ corresponds to a normal Markovian random walk.

By defining the displacement as $x_t \equiv X_t - X_0$, it has been shown that the first and the second moments of x_t are given by [20]:

$$\langle x_t\rangle = \beta\frac{\Gamma(t+\alpha)}{\Gamma(1+\alpha)\Gamma(t)}\,, \quad (2)$$

and

$$\langle x_t^2\rangle = \frac{t}{2\alpha-1}\left(\frac{\Gamma(t+2\alpha)}{\Gamma(t+1)\Gamma(2\alpha)}-1\right), \quad (3)$$

in which $\beta = 2q-1$. For $t \gg 1$ these quantities have the following asymptotic behaviors [20]:

$$\langle x_t\rangle \sim \frac{\beta}{\Gamma(1+\alpha)}t^{\alpha}\,, \quad (4)$$

and

$$\langle x_t^2\rangle \sim \begin{cases} \frac{t}{1-2\alpha} & \text{for} \quad \alpha<\frac{1}{2} \\ t\ln t & \text{for} \quad \alpha=\frac{1}{2} \\ \frac{t^{2\alpha}}{(2\alpha-1)\Gamma(2\alpha)} & \text{for} \quad \alpha>\frac{1}{2} \end{cases}. \quad (5)$$

In the following section, we will add random memory resetting to the ERW defined above and investigate its effects on the statistics of the walker.

3. ERW under random memory resetting

In this paper we define random memory resetting as follows: every time a random resetting occurs the memory of the ERW is erased so that the number of previously taken steps in the elephant's memory vanishes (A simple sketch is given in figure 1). Note that the first step of the ERW at $t=1$ could be a biased or an unbiased step, determined by q. We will call this the biased or the unbiased condition hereafter. After every random resetting, since there is no memory, the ERW takes a step according to the dynamical rule of the first step at $t=1$. In the presence of memory resetting the equation of motion can be obtained as follows. Similar to [20], in each step the elephant location can decrease or increase by one unit. The only difference here is that the probability of decreasing or increasing depends on the last resetting time; therefore, the location of the elephant evolves according to the following rule:

$$X_{t+1} = X_t + \sigma_{t+1}. \quad (6)$$

At time $t+1$, if memory resetting occurs, the elephant's memory is erased, and similar to the first step, σ_{t+1} takes the value 1 with the probability q, and -1 with the probability $(1-q)$. If memory resetting does not occur at time $t+1$, the elephant does not remember those steps taken before the time at which the last memory resetting happened. Now a time t' is randomly selected, with a uniform probability, between the last resetting time and t (if no memory resetting has occurred during the process, t' is randomly selected, with a uniform probability, between 1 and t). Then with the probability p, we have $\sigma_{t+1} = \sigma_{t'}$, while with the probability $(1-p)$, we have $\sigma_{t+1} = -\sigma_{t'}$. To calculate the ERW propagator in the presence of the resetting we can use the renewal equation to relate the ERW propagator with resetting P_r to that of without resetting P_0. The subscript r (0) indicates that this quantity is pertinent to the process in the presence (absence) of memory resetting. Using the renewal equation, the transition probability can be written as follows:

$$P_r(Y,t+1|X_0,0) = (1-r)^t P_0(Y,t+1|X_0,0) +$$

$\sum_{\tau=1}^{t}\sum_{J=-\infty}^{\infty} r(1-r)^{t-\tau} P_r(J,\tau|X_{0,}0)$
$P_0(Y,t+1|J,\tau)$. (7)

The first term refers to those trajectories in which no memory resetting occurrs where $(1-r)^t$ is the probability that no memory resetting has happened up to the time $t+1$. The second term refers to those trajectories in which memory resetting occurrs. $r(1-r)^{t-\tau}$ is the probability that the last memory resetting has happened at the time $\tau+1$ where τ is a time step between 1 and t. In (7), if $\tau \neq t$, $P_0(Y,t+1|J,\tau)$ should be obtained from (1) and for $\tau = t$ it is given by:

$$P_0(Y,t+1|J,\tau) = q\delta_{J,Y-1} + (1-q)\delta_{J,Y+1}.$$

4. Moments of displacement

Using (7) and after some calculations one finds the following recursion relation for the nth moment of x_t in the presence of random resetting at the time $t+1$ in terms of its value in previous times:

$$\langle x_{t+1}^n\rangle_r = \sum_{m=0}^{n}\binom{n}{m}\sum_{\tau=1}^{t} r(1-r)^{t-\tau}\langle x_\tau^m\rangle_r\langle x_{t-\tau+1}^{n-m}\rangle_0 + (1-r)^t\langle x_{t+1}^n\rangle_0 \ . \quad (8)$$

The proof is given in appendix A. As can be seen, to obtain the nth moment, we need all moments in the presence and absence of resetting of all times before $t+1$. Using (8) and after some lengthy but straightforward simplifications, one arrives at (the details are brought in appendix B):

$$\langle x_t^n\rangle_r = (1-r)^{t-1}\langle x_t^n\rangle_0 + 2r\sum_{\tau=1}^{t-1}(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + r\sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{t-1}(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{t-\tau}^m\rangle_r + r^2\sum_{\tau=1}^{t-2}(1-r)^{\tau-1}(t-\tau-1)\langle x_\tau^n\rangle_0 + r^2\sum_{m=1}^{n-1}\binom{n}{m}\sum_{k=1}^{t-2}(1-r)^{k-1}\langle x_k^{n-m}\rangle_0\sum_{\tau=1}^{t-k-1}\langle x_\tau^m\rangle_r \ . \quad (9)$$

Now for the first moment we have:

$$\langle x_t\rangle_r = (1-r)^{t-1}\langle x_t\rangle_0 + 2r\sum_{\tau=1}^{t-1}(1-r)^{\tau-1}\langle x_\tau\rangle_0 + r^2\sum_{\tau=1}^{t-2}(1-r)^{\tau-1}(t-\tau-1)\langle x_\tau\rangle_0. \quad (10)$$

It is easy to check that $\langle x_t\rangle_{r=0}$ results in $\langle x_t\rangle_0$ as expected. By substituting (2) in (10) we find:

$$\langle x_t\rangle_r = \beta r^{-\alpha}(-\alpha + r(\alpha+t-1)+1) + \frac{1}{\Gamma(\alpha+1)\Gamma(t)}(\beta(1-r)^{t-1}\Gamma(t+\alpha) (r^2\ {}_2F_1(2,t+\alpha;t;1-r) - 2r\ {}_2F_1(1,t+\alpha;t;1-r)+1), \quad (11)$$

in which the function ${}_2F_1(a,b;c;z)$ is the hypergeometric function defined as:

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty}\frac{z^k(a)_k(b)_k}{k!(c)_k}z^k,$$

where $(a)_k$ is the Pochhammer symbol. For $r \neq 0$ and large t, the second term in (11) drops to zero very quickly. In this case, the asymptotic behavior of the first moment can be readily calculated and is given by:

$$\langle x_t\rangle_r \sim \beta r^{1-\alpha}t \quad \text{for} \quad r\neq 0 \quad \text{and} \quad t \gg 1. \quad (12)$$

As can be seen from (4), for $\alpha < 0$ (a reformer ERW) the first moment of displacement goes to zero in the long-time limit; however, this is not the case in the presence of resetting as it grows linearly with t; therefore, the mean displacement increases indefinitely. Note that for $\beta = 0$ both (4) and (12) become zero. Finally, the direction of the escape from the starting position is determined by β as in the case $r = 0$.

We can use (9) to calculate the second moment of displacement in the presence of memory resetting:

$$\langle x_t^2\rangle_r = (1-r)^{t-1}\langle x_t^2\rangle_0 + 2r\sum_{\tau=1}^{t-1}(1-r)^{\tau-1}\langle x_\tau^2\rangle_0 + 2r\sum_{\tau=1}^{t-1}(1-r)^{\tau-1}\langle x_\tau\rangle_0\langle x_{t-\tau}\rangle_r + r^2\sum_{\tau=1}^{t-2}(1-r)^{\tau-1}(t-\tau-1)\langle x_\tau^2\rangle_0 + 2r^2\sum_{k=1}^{t-2}(1-r)^{k-1}\langle x_k\rangle_0\sum_{\tau=1}^{t-k-1}\langle x_\tau\rangle_r. \quad (13)$$

As in the previous case, it is easy to check that (13) gives the desired result at $r = 0$.

By substituting (2), (3), and (11) in (13) one can, in principle, calculate the exact analytical expression for $\langle x_t^2\rangle_r$; however, the result will be very complicated and does not have a closed form. For $\beta = 0$ the second moment in the presence of resetting $\langle x_t^2\rangle_r|_{\beta=0}$ will be given by the following exact expression obtained using MATHEMATICA software:

$$\langle x_t^2\rangle_r|_{\beta=0} = \frac{t-2\alpha r^{-2\alpha}(-2\alpha+r(2\alpha+t-1)+1)}{1-2\alpha} + (1-r)^{t-1}\Big(\frac{\Gamma(t+2\alpha)}{(1-2\alpha)\Gamma(2\alpha)}\big(r\,(t-1)\ {}_2\tilde{F}_1(1,t+2\,\alpha;t+1;1-r) + r(2\alpha(r-1)+1)\ {}_2\tilde{F}_1(2,t+2\alpha;t+1;1-r) - \frac{1}{\Gamma(t)}\big)\Big). \quad (14)$$

where ${}_2\tilde{F}_1(a,b\,;c;z)$ is the regularized hypergeometric function defined as ${}_2\tilde{F}_1(a,b\,;c;z) = {}_2F_1(a,b;c;z)/\Gamma(c)$. On the other hand, for $\beta \neq 0$ we find, using MATHEMATICA software, that (13) can be very well approximated by the following closed formula:

$$\langle x_t^2\rangle_r \approx \frac{t-2\alpha r^{-2\alpha}(-2\alpha+r(2\alpha+t-1)+1)}{1-2\alpha} + \beta^2 r^{-2\alpha}(3\alpha^2(r-1)^2+\alpha(r-1)(r(4t-5)+1) + r(t-1)(r(t-2)+2)) + (1-r)^{t-1}\Big(\frac{\Gamma(t+2\alpha)}{(1-2\alpha)\Gamma(2\alpha)}\big(r(t-1)\ {}_2\tilde{F}_1(1,t+2\alpha;t+1;1-r) + r(2\alpha(r-1)+1)\ {}_2\tilde{F}_1(2,t+2\alpha;t+1;1-r) - \frac{1}{\Gamma(t)}\big) + \frac{\beta^2 r^{-2\alpha}\Gamma(t+\alpha)}{\Gamma(\alpha+1)}\big((-3\,\alpha + r\,(3\,\alpha+t-2)+2)\ {}_2\tilde{F}_1(t-2,-\alpha;t;1-r) - (-2\alpha + r(2\alpha+t-2)+2)\ {}_2\tilde{F}_1(t-1,-\alpha;t;1-r)\big)\Big). \quad (15)$$

In the presence of resetting and the long-time limit the third term in (15) vanishes very quickly; therefore, its asymptotic behavior is given by:

$$\langle x_t^2\rangle_r \sim \begin{cases} \frac{1-2\alpha r^{1-2\alpha}}{1-2\alpha}t & \text{for} \quad \beta = 0 \\ \beta^2 r^{2-2\alpha}t^2 & \text{for} \quad \beta \neq 0 \end{cases}. \quad (16)$$

Surprisingly, for $r \neq 0$ the asymptotic behavior of (15) reduces to two different behaviors given in (16) depending on different values of β. In other words, the process is influenced by randomness (the first step after each reset) rather than memory dependence. If $\beta = 0$ the MSD grows diffusively with a diffusion coefficient $D = (1-2\alpha r^{1-2\alpha})/(2-4\alpha)$. In figure 2 (top) we have plotted (13) and, its asymptotic behavior for $r = 0.5$, $\beta = 0.9$ and, three values of α. We have also plotted (13) for $\beta = 0$, $\alpha = 0.5$ and, two values of r (bottom). As can be seen each curve (denoted by a dotted or a dashed line) and

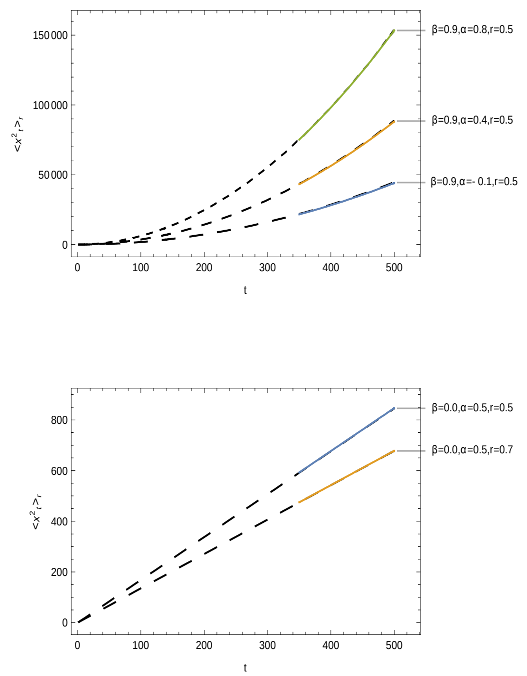

Figure 2. (Color Online) A plot of (13) and (16) as dashed and solid lines respectively for $r = 0.5, 0.7$ and, different values of α and β. It can be seen that the exact results and their asymptotics overlap in the long-time limit.

$r=0, \frac{1}{2} \leq \alpha \leq 1, -1 \leq \beta \leq 1$ Superdiffusive	$r=0, -1 \leq \alpha < \frac{1}{2}, -1 \leq \beta \leq 1$ Diffusive
$r \neq 0, -1 \leq \alpha \leq 1, \beta \neq 0$ Superdiffusive	$r \neq 0, -1 \leq \alpha \leq 1, \beta = 0$ Diffusive

Figure 3. The phase diagram of the process. It can be seen that the process might undergo a transition from diffusive to superdiffusive or vice versa depending on the values of the parameters α, β and r.

its asymptotic (denoted by a solid line) obtained from (16) lie along each other. In this case, regardless of the value of r, we have a diffusive behavior.

It can also be verified that in the presence of resetting, $\alpha = 0.5$ does not determine the transition between diffusive and superdiffusive behaviors which is in contrast to the case of $r = 0$. One should also note that turning the resetting on or off might result in a transition from a diffusive to a superdiffusive behavior or vice versa. For instance, starting with the values $r = 0$, $\alpha < 1/2$ and $\beta \neq 0$ and, then turning the resetting on, the process undergoes a transition from diffusive to superdiffusive behavior. The phase diagram of the model is given in figure 3.

5. Distribution of displacement under unbiased condition

In the absence of resetting, the PDF of the displacement of the walker $P_0(x,t)$ does not have a Gaussian distribution for $\alpha > 1/2$ regardless of the value of β [28,29].

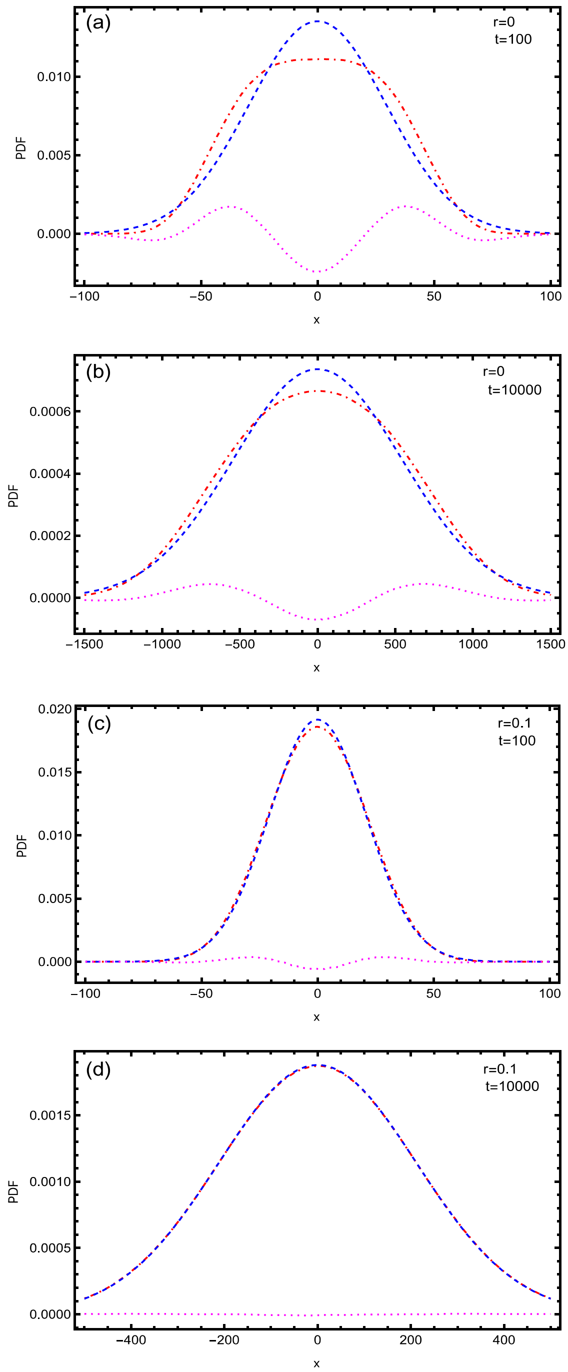

Figure 4. (Color Online) The graph illustrates the PDF of displacement under conditions with and without resetting at specific time points ($t = 100$ and 10000). We have considered $\beta = 0$ and $\alpha = 0.6$. Each plot includes the results of a Monte Carlo simulation (represented by a dot-dashed line) and a normal distribution (represented by a dashed line) with a mean of zero and a variance derived from equation (14). In the case of $r = 0$, depicted in figures (a) and (b), it is evident that the curves never overlap over time. Conversely, for $r = 0.1$, shown in figures (c) and (d), the curves progressively overlap, indicating a convergence towards a residual value of zero (represented by a dotted line).

In what follows we analytically show that, in the presence of resetting and under the unbiased condition $\beta = 0$, while the skewness is always zero, the kurtosis approaches zero in the long-time limit. In this case, using (9), it can be seen that all the odd moments are zero.

Because for $r = 0$ all the odd moments are zero, one can see that the third moment is zero under the unbiased condition:

$$\langle x_t^3\rangle_r = (1-r)^{t-1}\langle x_t^3\rangle_0 + 2r\sum_{\tau=1}^{t-1}(1-r)^{\tau-1}\langle x_\tau^3\rangle_0 + r\sum_{m=1}^{2}\binom{3}{m}\sum_{\tau=1}^{t-1}(1-r)^{\tau-1}\langle x_\tau^{3-m}\rangle_0\langle x_{t-\tau}^m\rangle_r + r^2\sum_{\tau=1}^{t-2}(1-r)^{\tau-1}(t-\tau-1)\langle x_\tau^3\rangle_0 + r^2\sum_{m=1}^{2}\binom{3}{m}\sum_{k=1}^{t-2}(1-r)^{k-1}\langle x_k^{3-m}\rangle_0\sum_{\tau=1}^{t-k-1}\langle x_\tau^m\rangle_r. \quad (17)$$

It is not difficult to check that all the terms in the above equation are equal to zero. One can do the same calculations for all the odd moments.

The above result leads to a zero skewness defined as:

$$\gamma_1(t) = \frac{\langle (x_t-\langle x_t\rangle)^3\rangle}{\langle (x_t-\langle x_t\rangle)^2\rangle^{\frac{3}{2}}}.$$

The kurtosis defined as:

$$\gamma_2(t) = \frac{\langle (x_t-\langle x_t\rangle)^4\rangle}{\langle (x_t-\langle x_t\rangle)^2\rangle^2} - 3$$

will then be simplified to:

$$\gamma_2(t) = \frac{\langle x_t^4\rangle}{\langle x_t^2\rangle^2} - 3. \quad (18)$$

We find that the fourth moment, according to (9), is given by:

$$\langle x_t^4\rangle_r = (1-r)^{t-1}\langle x_t^4\rangle_0 + 2r\sum_{\tau=1}^{t-1}(1-r)^{\tau-1}\langle x_\tau^4\rangle_0 + 6r\sum_{\tau=1}^{t-1}(1-r)^{\tau-1}\langle x_\tau^2\rangle_0\langle x_{t-\tau}^2\rangle_r + r^2\sum_{\tau=1}^{t-2}(1-r)^{\tau-1}(t-\tau-1)\langle x_\tau^4\rangle_0 + 6r^2\sum_{k=1}^{t-2}(1-r)^{k-1}\langle x_k^2\rangle_0\sum_{\tau=1}^{t-k-1}\langle x_\tau^2\rangle_r. \quad (19)$$

In the long time limit, the fifth term is dominant; therefore, the fourth moment behaves asymptotically as:

$$\langle x_t^4\rangle_r \sim \frac{3\left(1-2\alpha r^{1-2\alpha}\right)^2 t^2}{(1-2\alpha)^2}. \quad (20)$$

Now using (16), (18) and (20) one can see that in the long-time limit the kurtosis $\gamma_2$vanishes. This confirms that the PDF of the displacement becomes a Gaussian in the long-time limit.

In figure 4 we have plotted the Monte Carlo simulation results for the PDF of the displacement besides the analytical results. We have also brought the residual plot defined as the difference between the simulation data and those obtained analytically. The Gaussian PDF has a zero mean and its variance is obtained from (13). It turns out that the difference in the numerical results with the Gaussian PDF decreases as time increases in the presence of resetting. It can be shown that if $\beta \neq 0$ (the biased conditions), the skewness is not zero, so the PDF will not be Gaussian.

6. Concluding remarks

This paper investigates a non-Markovian discrete-time random walk model on an infinite one-dimensional lattice, incorporating memory resetting. Each instance of resetting results in the random walker completely erasing its entire memory. Our findings demonstrate that, subject to a specific condition referred to as the unbiased condition, wherein the initial step of the random walk following a reset is unbiased, the probability density function governing displacement follows a normal distribution. Furthermore, our analysis reveals that by toggling the resetting mechanism, the random walker exhibits a transition from diffusive behavior to superdiffusive behavior, or vice versa. Our analytical results are corroborated through the utilization of Monte Carlo simulations. Asymptotic analysis of the first and second moments of the displacement reveals that, unlike resetting to a specific point, memory resetting does not induce a non-equilibrium steady state.

Appendix A

To prove (8) we start by multiplying (7) in $(\boldsymbol{Y}-\boldsymbol{X_0})^{\boldsymbol{n}}$ and then sum over all values of $\boldsymbol{Y}$ which gives:

$$\sum_{Y=-\infty}^{\infty}(Y-X_0)^n P_r(Y,t+1|X_0,0) = \sum_{Y=-\infty}^{\infty}(Y-X_0)^n (1-r)^t P_0(Y,t+1|X_0,0) + \sum_{Y=-\infty}^{\infty}(Y-X_0)^n\sum_{\tau=1}^{t}\sum_{J=-\infty}^{\infty} r(1-r)^{t-\tau} P_r\left(J,\tau\middle|X_{0,}0\right)P_0(Y,t+1|J,\tau).$$

Using the fact that:

$$\boldsymbol{\sum_{Y=-\infty}^{\infty}(Y-X_0)^n P_r(Y,t+1|X_0,0) = \langle x_{t+1}^n\rangle_r,}$$

and:

$$\sum_{Y=-\infty}^{\infty}(Y-X_0)^n (1-r)^t P_0(Y,t+1|X_0,0) = (1-r)^t\langle x_{t+1}^n\rangle_0,$$

besides the following expansion:

$$(Y-X_0)^n = (Y-J+J-X_0)^n = \sum_{m=0}^{n}\binom{n}{m}(Y-J)^{n-m}(J-X_0)^m,$$

one can simplify the second term of the left- hand side as follows:

$$\sum_{Y=-\infty}^{\infty}(Y-X_0)^n\sum_{\tau=1}^{t}\sum_{J=-\infty}^{\infty} r(1-r)^{t-\tau} P_r\left(J,\tau\middle|X_{0,}0\right)P_0(Y,t+1|J,\tau) = \sum_{Y=-\infty}^{\infty}\sum_{m=0}^{n}\binom{n}{m}(Y-J)^{n-m}(J-X_0)^m\sum_{\tau=1}^{t}\sum_{J=-\infty}^{\infty} r(1-r)^{t-\tau} P_r\left(J,\tau\middle|X_{0,}0\right)P_0(Y,t+1|J,\tau) = \sum_{m=0}^{n}\binom{n}{m}\sum_{\tau=1}^{t} r(1-r)^{t-\tau}\sum_{J=-\infty}^{\infty}(J-X_0)^m P_r\left(J,\tau\middle|X_{0,}0\right)\sum_{Y=-\infty}^{\infty}(Y-J)^{n-m} P_0(Y,t+1|J,\tau) = \sum_{m=0}^{n}\binom{n}{m}\sum_{\tau=1}^{t} r(1-r)^{t-\tau}\langle x_{t-\tau+1}^{n-m}\rangle_0\sum_{J=-\infty}^{\infty}(J-X_0)^m P_r\left(J,\tau\middle|X_{0,}0\right) = \sum_{m=0}^{n}\binom{n}{m}\sum_{\tau=1}^{t} r(1-r)^{t-\tau}\langle x_{t-\tau+1}^{n-m}\rangle_0\langle x_\tau^m\rangle_r.$$

This will give the final result in (8).

Appendix B

To prove (9) we start from (8):

$\langle x_{t+1}^n\rangle_r = \sum_{m=0}^n \binom{n}{m} \sum_{\tau=1}^t r(1-r)^{t-\tau}\langle x_\tau^m\rangle_r \langle x_{t-\tau+1}^{n-m}\rangle_0 + (1-r)^t\langle x_{t+1}^n\rangle_0$,

to find:

$\langle x_t^n\rangle_r = \sum_{m=0}^n \binom{n}{m} \sum_{\tau=1}^{t-1} r(1-r)^{t-1-\tau}\langle x_\tau^m\rangle_r \langle x_{t-\tau}^{n-m}\rangle_0 + (1-r)^{t-1}\langle x_t^n\rangle_0$.

Expanding the above expression gives:

$\langle x_t^n\rangle_r = (1-r)^{t-1}\langle x_t^n\rangle_0 + \sum_{\tau=1}^{t-1} r(1-r)^{t-\tau-1}\langle x_{t-\tau}^n\rangle_0 + \sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{t-1} r(1-r)^{t-\tau-1}\langle x_\tau^m\rangle_r\langle x_{t-\tau}^{n-m}\rangle_0 + \sum_{\tau=1}^{t-1} r(1-r)^{t-\tau-1}\langle x_\tau^n\rangle_r$

$= (1-r)^{t-1}\langle x_t^n\rangle_0 + \sum_{\tau=1}^{t-1} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + \sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{t-1} r(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{t-\tau}^m\rangle_r + \sum_{\tau=1}^{t-1} r(1-r)^{t-\tau-1}\langle x_\tau^n\rangle_r$

.

We can write the above expression for different values of t. Here we bring the results up to $t=4$:

$\langle x_1^n\rangle_r = \langle x_1^n\rangle_0$,

$\langle x_2^n\rangle_r = (1-r)\langle x_2^n\rangle_0 + \sum_{\tau=1}^{1} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + \sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{1} r(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{2-\tau}^m\rangle_r + r\langle x_1^n\rangle_0$,

$\langle x_3^n\rangle_r = (1-r)^2\langle x_3^n\rangle_0 + \sum_{\tau=1}^{2} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + \sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{2} r(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{3-\tau}^m\rangle_r + r(1-r)\langle x_1^n\rangle_0$

$+r\left((1-r)\langle x_2^n\rangle_0 + \sum_{\tau=1}^{1} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + \sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{1} r(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{2-\tau}^m\rangle_r + r\langle x_1^n\rangle_0\right)$

$= (1-r)^2\langle x_3^n\rangle_0 + \sum_{\tau=1}^{2} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + \sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{2} r(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{3-\tau}^m\rangle_r + r\langle x_1^n\rangle_0$

$+r\left((1-r)\langle x_2^n\rangle_0 + \sum_{\tau=1}^{1} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + \sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{1} r(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{2-\tau}^m\rangle_r\right)$,

and:

$\langle x_4^n\rangle_r = (1-r)^3\langle x_4^n\rangle_0 + \sum_{\tau=1}^{3} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + \sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{3} r(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{4-\tau}^m\rangle_r + r(1-r)^2\langle x_1^n\rangle_0$

$r(1-r)\left((1-r)\langle x_2^n\rangle_0 + \sum_{\tau=1}^{1} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + \sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{1} r(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{2-\tau}^m\rangle_r + r\langle x_1^n\rangle_0\right)$

$+r\Big((1-r)^2\langle x_3^n\rangle_0 + \sum_{\tau=1}^{2} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + \sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{2} r(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{3-\tau}^m\rangle_r + r\langle x_1^n\rangle_0 + r\Big((1-r)\langle x_2^n\rangle_0 + \sum_{\tau=1}^{1} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + \sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{1} r(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{2-\tau}^m\rangle_r\Big)\Big)$

$= (1-r)^3\langle x_4^n\rangle_0 + \sum_{\tau=1}^{3} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + \sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{3} r(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{4-\tau}^m\rangle_r + r\langle x_1^n\rangle_0$

$+r\left((1-r)\langle x_2^n\rangle_0 + \sum_{\tau=1}^{1} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + \sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{1} r(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{2-\tau}^m\rangle_r\right)$

$+r\left((1-r)^2\langle x_3^n\rangle_0 + \sum_{\tau=1}^{2} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + \sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{2} r(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{3-\tau}^m\rangle_r\right)$.

Now, by intuition, one finds:

$\langle x_t^n\rangle_r = (1-r)^{t-1}\langle x_t^n\rangle_0 + \sum_{\tau=1}^{t-1} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + \sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{t-1} r(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{t-\tau}^m\rangle_r + r\langle x_1^n\rangle_0$

$\sum_{k=1}^{t-2} r\left((1-r)^k\langle x_{k+1}^n\rangle_0 + \sum_{\tau=1}^{k} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + \sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{k} r(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{k-\tau+1}^m\rangle_r\right)$.

By reorganizing and simplifying some of the terms in the above sum such as:

$r\langle x_1^n\rangle_0 + \sum_{k=1}^{t-2} r(1-r)^k\langle x_{k+1}^n\rangle_0 + \sum_{\tau=1}^{t-1} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 = 2r\sum_{\tau=1}^{t-1}(1-r)^{\tau-1}\langle x_\tau^n\rangle_0$,

$\sum_{k=1}^{t-2} r\sum_{\tau=1}^{k} r(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 = r^2\sum_{\tau=1}^{t-2}(1-r)^{\tau-1}(t-\tau-1)\langle x_\tau^n\rangle_0$,

and:

$\sum_{k=1}^{t-2} r\sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{k} r(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{k-\tau+1}^m\rangle_r = r^2\sum_{m=1}^{n-1}\binom{n}{m}\sum_{k=1}^{t-2}(1-r)^{k-1}\langle x_k^{n-m}\rangle_0\sum_{\tau=1}^{t-k-1}\langle x_\tau^m\rangle_r$,

we can rewrite $\langle x_t^n\rangle_r$ as follows to obtain (9):

$\langle x_t^n\rangle_r = (1-r)^{t-1}\langle x_t^n\rangle_0 + 2r\sum_{\tau=1}^{t-1}(1-r)^{\tau-1}\langle x_\tau^n\rangle_0 + r\sum_{m=1}^{n-1}\binom{n}{m}\sum_{\tau=1}^{t-1}(1-r)^{\tau-1}\langle x_\tau^{n-m}\rangle_0\langle x_{t-\tau}^m\rangle_r$

$+r^2\sum_{\tau=1}^{t-2}(1-r)^{\tau-1}(t-\tau-1)\langle x_\tau^n\rangle_0 + r^2\sum_{\tau=1}^{t-2}\sum_{\tau=1}^{k}(1-r)^{\tau-1}(t-\tau-1)\langle x_\tau^n\rangle_0$.

References

1. C D Bello, A K Hartmann, S N Majumdar, F Mori, A Rosso and G Schehr, *Phys. Rev. E* **108** (2023) 014112.
2. J Villarroel, M Montero, J A Vega, *Entropy* **23, 7** (2021) 825.
3. A Pal, R Chatterjee, S Reuveni, and A Kundu, *J. Phys. A: Math. Theor.* **52** (2019) 264002.
4. R K Singh, K Górska and T Sandev, *Phys. Rev. E* **105** (2022) 064133.
5. M R Evans, S N Majumdar, and G Schehr, *J. Phys. A Math. Theor.* **53** (2020) 193001.
6. H Chen and F Huang, *Phys. Rev. E* **105** (2022) 034109.
7. O T Friedman, Y Roichman and S Reuven, *Phys. Rev. E* **106(5)** (2022) 054116.
8. C D Bello, A V Chechkin, A K Hartmann, Z Palmowski, and R Metzler, *New J. Phys.* **25** (2023) 082002.
9. S N Majumdar, S Sabhapandit, and G Schehr, *Phys. Rev. E* **92** (2015) 052126.
10. D Boyer and C Solis-Salas, *Phys. Rev. Lett.* **112** (2014) 240601.
11. O Vilka, D Camposd, V Méndezd, E Lourieb, R Nathanb and M Assafa, *Phys. Rev. Lett.* **128** (2022) 148301.
12. A Stanislavsky and A Weron, *Phys. Rev. E* **104** (2021) 014125.
13. P Grigolini, *Chaos, Solitons & Fractals* **81** (2015) 575.
14. I Bronstein, Y Israel, E Kepten, S Mai, Y Shav-Tal, E Barkai and Y Garini, *Phys. Rev. Lett.* **103** (2009) 018102.
15. E Kepten, I Bronshtein, and Y Garini, *Phys. Rev. E* **87** (2013) 052713.

16. J Szymanski and M Weiss, *Phys. Rev. Lett.* **103** (2009) 038102.
17. D Brockmann, L Hufnagel, and T Geisel, *Nature* **439** (2006) 462.
18. R N Mantegna and H E Stanley, "*Introduction to Econophysics: Correlations and Complexity in Finance*", Cambridge University Press, Cambridge, (2000).
19. J-P Bouchaud and M Potters, "Theory of Financial Risk and Derivative Pricing", Cambridge University Press, Cambridge, (2000).
20. G M Schütz and S Trimper, *Phys. Rev. E* **70** (2004) 045101(R).
21. J C Cressoni, G M Viswanathan, and M A A da Silva, *J. Phys. A: Math. Theor.* **46** (2013) 505002.
22. J C Cressoni, M A A da Silva, and G M Viswanathan, *Phys. Rev. Lett.* **98** (2007) 070603.
23. H-J Kim, *Phys. Rev. E* **90** (2014) 012103.
24. U Harbola, N Kumar, and K Lindenberg, *Phys. Rev. E* **90** (2014) 022136.
25. N Kumar, U Harbola, and K Lindenberg, *Phys. Rev. E* **82** (2010) 021101.
26. M A A da Silva, G Viswanathan, and J Cressoni, *Physica A: Statistical Mechanics and its Applications* **421** (2015) 522.
27. J C Cressoni, G M Viswanathan, A S Ferreira, and M A A da Silva, *Phys. Rev. E* **86** (2012) 022103.
28. M A A da Silva, J C Cressoni, G M Schütz And G M Viswanathan and S Trimper, *Phys. Rev. E* **88** (2013) 022115.
29. F N C Paraan and J P Esguerra, *Phys. Rev. E* **74** (2006) 032101.